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Example where there is a martingale deflator but no equivalent martingale measure

Here is an example that shows that there might be no arbitrage but there does not exist an equivalent martingale measure over the infinite horizon.

Let $\xi_1, \xi_2, \ldots$ be independent random variables with

$$\mathbb{P}(\xi_i = 1) = p = 1 - q = \mathbb{P}(\xi_i = -1)$$

and let $S_t = \xi_1 + \ldots + \xi_t$ be a simple random walk, where we assume that it is not symmetrical $p \neq q$.

Define a market model a two asset model with respect to the natural filtration $\mathcal{F}_t = \sigma(\xi_1, \ldots, \xi_t)$ by

$$P_t = (1, S_t).$$

In particular, there is a numéraire with constant price $N_t = 1$, which can be interpreted as cash.

First let us compute all martingale deflators for the model. Fix t and $\xi_1, \ldots, \xi_t$ and let

$$Z_u = Y_{t+1}/Y_t \text{ if } \xi_{t+1} = 1, \text{ and } Z_d = Y_{t+1}/Y_t \text{ if } \xi_{t+1} = -1.$$

Since PY is a martingale, we have

$$\begin{aligned} &Y_t Z_u p + Y_t Z_d q = Y_t \\ &(S_t + 1) Y_t Z_u p + (S_t - 1) Y_t Z_d q = S_t Y_t \end{aligned}$$

so that $Z_u = 1/(2p)$ and $Z_d = 1/(2q)$. Hence, we have shown that all martingale deflators satisfy

$$Y_{t+1} = Y_t (4pq)^{-1/2} (q/p)^{\xi_{t+1}/2}$$

and hence

$$Y_t = Y_0 (4pq)^{-t/2} (q/p)^{S_t}.$$

Now fix a horizon $T > 0$ and let $\mathbb{P}_T$ be the restriction of $\mathbb{P}$ to $\mathcal{F}_T$. Let $\mathbb{Q}_T$ be the equivalent measure on $\mathcal{F}_T$ with density

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = Y_T / Y_0.$$

By the above discussion, $\mathbb{Q}_T$ is the equivalent martingale measure for the finite horizon model $(P_t)_{0 \leq t \leq T}$.

We now show that the measure $\mathbb{Q}_T$, the random variables $\xi_1, \ldots, \xi_T$ are independent with

$$\mathbb{Q}_T(\xi_i = 1) = \frac{1}{2} = \mathbb{Q}_T(\xi_i = -1).$$

Indeed, given $x_1, \ldots, x_T$, where $x_i \in \{-1, +1\}$ we have

$$\begin{aligned} \mathbb{Q}_T(\xi_1 = x_1, \ldots, \xi_T = x_T) &= \mathbb{E}^{\mathbb{P}_T}\left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \mathbb{1}_{\{\xi_1 = x_1, \ldots, \xi_T = x_T\}}\right] \\ &= \mathbb{E}^{\mathbb{P}_T}\left[(4pq)^{-T/2} (q/p)^{S_T} \mathbb{1}_{\{\xi_1 = x_1, \ldots, \xi_T = x_T\}}\right] \\ &= (4pq)^{-T/2} (q/p)^{(u-d)/2} p^u q^d \\ &= 2^{-T} \end{aligned}$$

where u is the number of indices i such that $x_i = +1$ and $d = T - u$ is the number of i such that $x_i = -1$.

Let us consider the measure $\mathbb{Q}$ on $\mathcal{F}$ with the property that the random variables $\xi_1, \xi_2, \ldots$ are independent with

$$\mathbb{Q}(\xi_i = 1) = \frac{1}{2} = \mathbb{Q}(\xi_i = -1),$$

so that $\mathbb{Q}_T$ is the restriction of $\mathbb{Q}$ to $\mathcal{F}_T$. Is this measure $\mathbb{Q}$ an equivalent martingale for the infinite horizon model $(P_t)_{t \geq 0}$? While it is true that P is a $\mathbb{Q}$-martingale, it is not true that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. Indeed,

$$\mathbb{P}\left(\frac{S_t}{t} \to p - q\right) = 1, \text{ but } \mathbb{Q}\left(\frac{S_t}{t} \to 0\right) = 1.$$

Since we have assumed $p \neq q$, we see that these measures are inequivalent! Indeed, note that

$$\mathbb{P}\left(Y_t \to 0\right) = 1, \text{ but } \mathbb{Q}\left(Y_t \to \infty\right) = 1.$$

Hence, the market has no arbitrage, yet there does not exist an equivalent martingale measure over the infinite horizon. (But there is an equivalent martingale measure $\mathbb{Q}_T$ over any finite horizon T, no matter how long.)