Ex 4.1

From $\underline{\Pi}$ and $\underline{X_0}$ possible to reconstruct $\underline{\varphi}$ by self-financing

$$\frac{dB_t}{B_t} = 2\,dt$$

$$\frac{dS^1}{S^1} = 3\,dt + dW_t^1 - 2\,dW_t^2$$

$$\frac{dS^2}{S^2} = 5\,dt - 2\,dW^1 + 4\,dW^2$$

$\Pi = ?$

$H \simeq (\varphi, \Pi)$

Bank, stocks

$$d\left(\frac{X}{B}\right) = \underline{\pi} \cdot d\left(\frac{S}{B}\right)$$

Itô's formula and self-financing

$\sim$ meas divide B

$$d\tilde{S}^1 = \underbrace{1}_{3-2}\,dt + dW^1 - 2\,dW^2$$

$$d\tilde{S}^2 = \underbrace{3}_{5-2}\,dt - 2\,dW^1 + 4\,dW^2$$

2

$$\boxed{dX = \underbrace{(\pi^1 + 3\pi^2)}\,dt + (\underline{\pi^1 - 2\pi^2})\,dW^1 + (-2\pi^1 + 4\pi^2)\,dW^2}$$

$$\pi^1 = 2\pi^2$$

What is Black-Scholes formula?

Price of a European call under the (minimal replicating cost) Black-Scholes model for asset prices (i.e. const interest rate and stock = geometric BM).

$$C_t(T,K) = S_t\,\Phi(d_1) - K e^{-r(T-t)}\,\Phi(d_2).$$

$$d_1 = \frac{-\log\left(Ke^{-r(T-t)}/S_t\right)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d\tilde{X} = \left(\frac{\pi^1}{\tilde{S}^1} + 3\frac{\pi^2}{\tilde{S}^2}\right)dt$$

$$+\left(\frac{\pi^1}{\tilde{S}^1} - 2\frac{\pi^2}{\tilde{S}^2}\right)dW^1$$

$$+\left(-2\frac{\pi^2}{\tilde{S}^1} + 4\frac{\pi^2}{\tilde{S}^2}\right)dW^2$$

How to build an arb:

~ divide by B

$$\boxed{\text{Set } \frac{\pi^1}{\tilde{S}^1} = 2\frac{\pi^2}{\tilde{S}^2} \Rightarrow \frac{\pi^1}{S^1} = 2\frac{\pi^2}{S^2}}$$

Ex 3.10(c)

Strict local martingales

Let X solve

$$dX = X^2\,dW, \quad W = BM.$$

Ex 3.9.

Set $\boxed{X_t = \dfrac{1}{\|Z_t - (1,0,0)\|}} = \dfrac{1}{\sqrt{(Z^1-1)^2 + (Z^2)^2 + (Z^3)^2}}$, $Z = $ 3-D BM.

Ito $dX = -\dfrac{(Z^1-1)\,dZ^1}{\left((Z^1-1)^2 + (Z^2)^2 + (Z^3)^2\right)^{3/2}}$

$= (Z^2\,dZ^2)X^3 - Z^3\,dZ^3\,X^3$

$$dX = -X^2 \boxed{\left(\frac{(z^1-1)\,dz^1 + z^2\,dz^2 + z^3\,dz^3}{\sqrt{(z^1-1)^2 + (z^2)^2 + (z^3)^2}} \right)}$$

Loc Mart Quadratic variation

dt, $W = \int_0^t$

W is a BM by Lévy characterisation

Ex. 3.9.

$$\cancel{dX = X^2\, dW}$$

$$\boxed{\mathbb{E}(X_t) < X_0}$$

X is a strict local mart.

Suppose that X was a true <u>martingale</u> $\to$ derive a contradiction

$$dX_t = X_t\,(X_t\,dW_t) \qquad \left(X_t > 0 \text{ a.s. } \forall t\right)$$

$$X_t = e^{-\int_0^t \frac{X_s^2}{2}\,ds + \int_0^t X_s\,dW_s}$$

Fix $T > 0$. Let $\mathbb{Q}$: $\frac{d\mathbb{Q}}{d\mathbb{P}} = X_T$.

$$\hat{W}_t = W_t - \int_0^t X_s\,ds$$

Girsanov theorem: $\hat{W}$ is a $\mathbb{Q}$-BM.

$$d\langle X\rangle = X^4\,dt$$

$$dX = X^2\,dW$$

$$= X^2(d\hat{W} + X\,dt).$$

Contradiction, on

Let $Y = \frac{1}{X}$.

$$dY = -\frac{1}{X^2}\,dX + \left(\frac{1}{2}\right)\left(\frac{2}{X^3}\right)d\langle X\rangle$$

$$= -(d\hat{W} + X\,dt) + \frac{1}{X^3}X^4\,dt$$

$$dY = -d\hat{W}.$$ a. BM under Q

$$P(Y_T > 0) = 1$$

$Y = 1/X$, $X > 0$ P.a.s.

$$Q(Y_T > 0)$$

$$Q(-\hat{W}_T + Y_0 > 0) < 1$$

3 (b) Sample.

Z adapted, integrable.

$U_T = Z_T$

$U_t = \max\{ Z_t, \mathbb{E}(U_{t+1} \mid \mathcal{F}_t) \}$

$S_t = S_0 + \xi_1 + \cdots + \xi_t$

$(\xi_i)_i$ are i.i.d. generate filtration.

$Z_t = f(S_t)$.

To show $\exists$ function V: $U_t = V(t, S_t)$

$$U_T = Z_T = f(S_T). \qquad V(T,S) = f(S) \quad \forall S.$$

Suppose $U_{t+1} = V(t+1, S_{t+1})$

$$U_t = \max\{Z_t, \mathbb{E}(U_{t+1} \mid \mathcal{F}_t)\}$$

$$= \max\{f(S_t), \mathbb{E}[V(t+1, S_t + \xi_{t+1}) \mid \mathcal{F}_t]\}$$

$$\int V(t+1, S_t + x)\, \mu(dx)$$

μ = law of ξ_{t+1}

$$U_t = V(t, S_t)$$

where $V(t,S) = \max\{f(S), \int V(t+1, S+x)\,\mu(dx)\}$

induction

Sample 9 (a)

$$dB = B\, r\, dt \qquad \left(B_t = e^{rt} \right)$$

$$dS = S\left(r\, dt + \sqrt{V}\, dW \right)$$

$$dV = (a - bV)\, dt + \underline{c\sqrt{V}}\left(\rho\, dW + \sqrt{1-\rho^2}\, dZ \right)$$

W, Z are indep BMs.

$$Z_t = F(t, S_t, V_t).$$

Where: $F(T, S, V) = \sqrt{S}$

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2} V S^2 \frac{\partial^2 F}{\partial S^2} + (a - bV)\frac{\partial F}{\partial V} + \frac{1}{2} c^2 V \frac{\partial^2 F}{\partial V^2} + c\rho V S \frac{\partial^2 F}{\partial V \partial S}$$

$$rF = 0$$

If there exists a risk-neutral measure (EMM relative to B)

then no arb relative to B.

Claim: $\mathbb{P}$ is risk-neutral

To show $\frac{S}{B}$ and $\frac{Z}{B}$ are loc marts.

$$d\left(\frac{S}{B}\right) = S\sqrt{v}\, dW$$

$$dZ$$

$$d\left(Z/B\right) = \frac{\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S} dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2} d\langle S\rangle + \frac{\partial F}{\partial v} dv + \frac{1}{2}\frac{\partial^2 F}{\partial v^2} d\langle v\rangle + \frac{\partial^2 F}{\partial v \partial S} d\langle S, v\rangle\right)}{B} - \frac{Z}{B} r\, dt$$

$$d\left(\frac{S}{B}\right) = d\left(e^{-rt} S\right) = -r e^{-rt} S\, dt + e^{-rt}\, dS$$

no dW term $\to$ ordinary product rule

$$= \boxed{e^{-rt}}\left(dS - rS\, dt\right)$$

$$= \text{stuff } dW + \text{other stuff } dZ + 0\, dt$$

$\Rightarrow \frac{S}{B}$ is a loc martingale

(stochastic integral with respect to BMs)

$\mathbb{Q}$ is a Equivalent (local) Martingale Measure relative to a numéraire N

means $\left(\frac{P_t}{N_t}\right)_t$ is a (local) martingale

Thm If $\exists$ E$\mathbb{Q}$mm Then no arb relative to numéraire.

Main idea: non-neg local martingales are supermartingales

→ (discrete time only: non-neg local martingales are true martingales)

Risk-neutral : ELMM relative to bank.

Ex 3.1(e)

Typo $\pi_t = \Pi(t, Z_{t-1})$.

(not $\Phi(t, Z_{t-1})$)

Ex 3.2 (c)

Heath-Jarrow-Morton (1993)

Continuous time.

Take forward rate curve as given. (unlike the models where spot rate is given)

$$1 + f_{t,T} = (1 + f_{t-1,T})\, \eta_{t,T} \qquad 1 \le t \le T$$

$$\to \mathbb{E}\left[\left(\prod_{u=t+1}^{T} \eta_{t,u} \right)^{-1} \middle| \mathcal{F}_{t-1} \right] = 1$$

$$\eta_{t,T} = e^{\sigma_{t,T} \varepsilon_t + \mu_{t,T}}, \qquad \sigma_{t,T},\ \mu_{t,T} \text{ are } \mathcal{F}_{t-1} \text{ meas}$$

$\xi_t \sim N(0,1)$, indep of $\mathcal{F}_{t-1}$

$$E\left(e^{-\sum_{u=t+1}^{T} \sigma_{t,u} \xi_t - \left(\sum_{u=t+1}^{T} \mu_{t,u}\right)} \,\Big|\, \mathcal{F}_{t-1} \right) = 1$$

$$\frac{1}{2}\left(\sum_{u=t+1}^{T} \sigma_{t,u}\right)^2 = \sum_{u=t+1}^{T} \mu_{t,u}$$

$$\frac{1}{2}\left(\sum_{u=t+1}^{T-1} \sigma_{t,u}\right)^2 = \sum_{u=t+1}^{T-1} \mu_{t,u}$$

$$\mu_{t,T} = \frac{1}{2}\left(\sum^{T-1} + \sigma_{t,T}\right)^2 - \frac{1}{2}\left(\sum^{T-1}\right)^2$$

$$= \left(\sum_{u=t+1}^{T-1} \sigma_{t,u}\right)\sigma_{t,T} + \frac{1}{2}\sigma_{t,T}^2$$

HJM.

$$df_{t,T} = \mu_{t,T}\,dt + \sigma_{t,T}\,dW_t$$

$$P_{t,T} = e^{-\int_t^T f_{t,u}\,du}, \quad f_{t,t} = r_t$$

HJM drift condition

$$\mu_{t,T} = \sigma_{t,T} \int_t^T \sigma_{t,u}\,du$$

$\rightarrow$ ensures that $\left(e^{-\int_0^t r_s\,ds} P_{t,T}\right)_{0 \le t \le T}$

is a loc martingale

(means meas in risk neutral $\tilde{=}$ L̃mm for bank) $\Rightarrow$ no arb rel to bank

Ex 3.6 (c)

$$X_t = e^{at} x + b \int_0^t e^{a(t-s)} \, dW_s .$$

$$dX = a X \, dt + b \, dW$$

dist of $\int_0^T X_t \, dt$

$t \mapsto X_t(\omega)$ is continuous.

$\int_0^T X_t \, dt$ is Riemann.

$$= \lim \boxed{\sum_{i=1}^{n} X_{t_i} \Delta t_i} \quad \text{a.s.}$$

$$X_t = x + \underline{a \int_0^t X_s \, ds} + \underline{b} W_t$$

$$\mathbb{E}\left(\int_0^t \left(\int_0^t f(s,u)\, ds\right)^2\right) < \infty$$

$$\int_0^t X_s\, ds = \frac{1}{a}\left(X_t - x - bW_t\right)$$

sufficient:
continuous
in (s,u)
not random

$$= \frac{1}{a}\left(\left(e^{at} - 1\right)x + b\int_0^t \left(e^{a(t-s)} - 1\right) dW_s\right)$$

$$\boxed{f(s,u)\, 1_{\{u \leq s\}}}$$

$$W_t = \int_0^t W_s\, ds$$

Take

$f = 1_{A \times B}$

A, B intervals

$$= \left(\frac{e^{at} - 1}{a}\right)x + b\underbrace{\int_0^t \left(\frac{e^{a(t-s)} - 1}{a}\right) dW_s}_{\text{non-random}},$$

$$b\int_0^t \left(\int_0^s e^{a(s-u)}\, dW_u\right) ds$$

$$\int_0^t \left(\int_0^t f(s,u)\, dW_u\right) ds \overset{?}{=} \int_0^t \left(\int_0^t f(s,u)\, ds\right) du$$

Ex 3.5

$$\Rightarrow \int_0^t f(s)\, dW_s \sim N\left(0, \int_0^t f(s)^2\, ds\right)$$