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Example sheet 5 - Michaelmas 2010

Problem 1. Suppose we make the reasonable assumptions that the bond prices are bounded $0 \leq P(t,T) \leq 1$ and that the interest rates are non-negative $r_t \geq 0$ for all $t \geq 0$. Show that if the process $(\tilde{P}(t,T))_{t\in[0,T]}$ is a local martingale, then it is actually a true martingale and, in particular, the bond prices are given by the formula

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right).$$

Problem 2. (Forward measure) Let $(r_t)_{t\in\mathbb{R}_+}$ be a continuous process such that $e^{-\int_0^T r_s ds}$ is $\mathbb{Q}$-integrable for all $T > 0$.

Suppose

$$P(t,\tau) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^\tau r_s ds}|\mathcal{F}_t)$$

for all $\tau \geq t$. For fixed $T > 0$, define a measure $\mathbb{Q}_T$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{e^{-\int_0^T r_s ds}}{P_0(T)}.$$

(1) Show that for all $\tau > 0$, the process $(P(t,\tau)/P(t,T))_{t\in[0,\tau\wedge T]}$ is a martingale for $\mathbb{Q}_T$.
(2) Show that the forward rate $(f(t,T))_{t\in[0,T]}$ is a martingale for $\mathbb{Q}_T$.
(3) Show that there is no arbitrage if the European claim with maturity T and bounded payout ξ_T has time-t price

$$\xi_t = P(t,T)\ \mathbb{E}^{\mathbb{Q}_T}(\xi_T|\mathcal{F}_t).$$

Problem 3. (Vasicek model) Let

$$dr_t = -\lambda(\bar{r} - r_t)\ dt + \sigma\ dW_t$$

for positive constants $\lambda, \bar{r}$, and σ. Show that

$$\int_0^t r_s\ ds \sim N\left(\bar{r}t + (r_0 - \bar{r})\frac{(1-e^{-\lambda t})}{\lambda}, \frac{\sigma^2}{\lambda^2}\int_0^t (1-e^{-\lambda s})^2\ ds\right).$$

Use the moment generating function of a normal random variable to compute

$$f(0,T) = e^{-\lambda T} r_0 + (1-e^{-\lambda T})\bar{r} - \frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda T})^2$$

Show that if $r_0 \geq \bar{r}$ then $T \mapsto f(0,T)$ is decreasing.

Problem 4. (Vasicek model) Fix a time horizon $S > 0$. Express the dynamics of the short rate in the Vasicek model in terms of the $\mathbb{Q}_S$-Brownian motion, where $\mathbb{Q}_S$ is the S-forward measure. Hence, deduce that the distribution of r_S under $\mathbb{Q}_S$ is

$$N\left(r_0 e^{-\lambda S} + (1-e^{-\lambda S})\bar{r} - \frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda S})^2, \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda S})\right).$$

Use the Vasicek model to compute the time-0 price of a European call option maturing at time S with strike K, written on a zero-coupon bond with maturity $T > S$.

Problem 5. (Hull–White extension of Cox–Ingersoll–Ross) Consider the short rate model given by

$$dr_t = -\lambda(\bar{r}(t) - r_t)\ dt + \sigma\sqrt{r_t}\ dW_t$$

for positive constants λ and σ and a deterministic function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}$. Find the initial forward rate curve $f_0 = f(0, \cdot)$ for this model. [Is it possible to find a $\bar{r}$ that exactly match a given f_0?]

Problem 6. (interest rate swap) Party A agrees to give Party B a stream of payments throughout the interval $t \in [0, T]$, such that during the infinitesimal interval $(t, t + dt)$ Party B receives $r_t\ dt$ units of money, where r_t is the short rate at time t. During the same period, Party B agrees to pay Party A $s\ dt$ units of money during the interval $(t, t + dt)$, where s is a fixed constant. If no money changes hands at time 0, to what value should the constant s be set so that there is no arbitrage? Your answer should be in terms of the time 0 bond prices $P(0, T)$. The constant $s = s(0, T)$ is called the swap rate for maturity T. Show that the left end point of the swap rate curve $s(0, \cdot)$ is the short rate r_0.

Problem 7. Show that there is no arbitrage if the dynamics of the forward rates are given by

$$df(t, T) = \cos(T - t)[\sin(T - t)dt + dW_t]$$

for a scalar Brownian motion $(W_t)_{t \in \mathbb{R}_+}$.

Problem 8. Let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{E}(X_n) = 1$ for all n. Use the Borel–Cantelli lemma to show

$$\limsup_{n \to \infty} X_n^{1/n} \le 1 \text{ a.s.}$$

If $y(t, T)$ is the yield of a bond maturing at time T, the *long rate* is defined as $\ell_t = \lim_{T \to \infty} y(t, T)$ whenever the limit exists. Assuming that the bonds are priced by expectation, show that the long rate is non-decreasing, that is

$$\ell_t \ge \ell_s \text{ a.s. for all } 0 \le s \le t,$$

a fact first discovered by Dybvig, Ingersoll, and Ross in 1996.