**Advanced Financial Models**
Example sheet 5 - Michaelmas 2010

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**Problem 1.** Suppose we make the reasonable assumptions that the bond prices are bounded $0 \leq P_t(T) \leq 1$ and that the interest rate is non-negative $r_t \geq 0$. Show that if the process $(\tilde{P}_t(T))_{t\in[0,T]}$ is a local martingale, then it is actually a true martingale and, in particular, the bond prices are given by the formula

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right).$$

*Solution* 1. The conclusion follows from the inequality $0 \leq \tilde{P}_t(T) = e^{-\int_0^t r_s ds} P_t(T) \leq 1$ and the fact that uniformly bounded local martingales are true martingales. See exercise 3.6.

**Problem 2.** (Forward measure) Let $(r_t)_{t\in\mathbb{R}_+}$ be a continuous process such that $e^{-\int_0^T r_s ds}$ is $\mathbb{Q}$-integrable for all $T > 0$.

Suppose

$$P(t,\tau) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^\tau r_s ds}|\mathcal{F}_t)$$

for all $\tau \geq t$. For fixed $T > 0$, define a measure $\mathbb{Q}_T$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{e^{-\int_0^T r_s ds}}{P_0(T)}.$$

(1) Show that for all $\tau > 0$, the process $(P(t,\tau)/P(t,T))_{t\in[0,\tau\wedge T]}$ is a martingale for $\mathbb{Q}_T$.
(2) Show that the forward rate $(f(t,T))_{t\in[0,T]}$ is a martingale for $\mathbb{Q}_T$.
(3) Show that there is no arbitrage if the European claim with maturity $T$ and bounded payout $\xi_T$ has time-$t$ price

$$\xi_t = P(t,T)\ \mathbb{E}^{\mathbb{Q}_T}(\xi_T|\mathcal{F}_t).$$

*Solution* 2. Remember: if $M_t = \mathbb{E}(M_T|\mathcal{F}_t)$ for all $t \in [0,T]$, then $(M_t)_{t\in[0,T]}$ is a martingale.

(1)

$$\begin{aligned}
\frac{P(t,\tau)}{P(t,T)} &= \frac{\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^\tau r_s ds}|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s\ ds}|\mathcal{F}_t)} \\
&= \frac{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds} e^{\int_\tau^T r_s ds}|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds}|\mathcal{F}_t)} \\
&= \mathbb{E}^{\mathbb{Q}_T}(e^{\int_\tau^T r_s ds}|\mathcal{F}_t)
\end{aligned}$$

(2)

$$\begin{aligned}
f(t,T) &= -\frac{\partial \log P(t,T)}{\partial T} \\
&= \frac{\mathbb{E}^{\mathbb{Q}}(r_T e^{-\int_0^T r_s\ ds}|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s\ ds}|\mathcal{F}_t)} \\
&= \mathbb{E}^{\mathbb{Q}_T}(r_T|\mathcal{F}_t)
\end{aligned}$$

The measure $\mathbb{Q}_T$ is called the $T$-*forward measure* as it is a measure under which the forward rate for maturity $T$ is a martingale.

(3) Our current market comprises all bonds of maturity $\tau$, for all $\tau > 0$, and the bank account $(B_t)_{t\in\mathbb{R}_+}$, where

$$B_t = e^{\int_0^t r_s ds}.$$

The idea is to let the bond of maturity $T$ be the numéraire. In part (1) we have shown that all of the bonds, when discounted by the bond of maturity $T$, are martingales for $\mathbb{Q}_T$. We now show that the discounted bank account is a martingale

$$\begin{aligned}\frac{B_t}{P_t(T)} &= \frac{e^{\int_0^t r_s ds}}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s\ ds}|\mathcal{F}_t)} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds}e^{\int_0^T r_s ds}|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds}|\mathcal{F}_t)} \\ &= \mathbb{E}^{\mathbb{Q}_T}(e^{\int_0^T r_s ds}|\mathcal{F}_t).\end{aligned}$$

Hence, all of these assets, when discounted by the bond of maturity $T$, are martingales under the forward measure $\mathbb{Q}_T$. Thus, there is no-arbitrage in the original market.

Now augment the market with a new asset $(\xi_t)_{t\in[0,T]}$. Since

$$\frac{\xi_t}{P(t,T)} = \mathbb{E}^{\mathbb{Q}_T}(\xi_T|\mathcal{F}_t)$$

is again a martingale for $\mathbb{Q}_T$, the augmented market has no arbitrage.

[You may prefer to work with the bank account as the numéraire, as we have usually done in lecture. In this case, the original market is free of arbitrage since

$$\frac{P(t,\tau)}{B_t} = \mathbb{E}^{\mathbb{Q}}(e^{-\int_0^\tau r_s ds}|\mathcal{F}_t)$$

defines a martingale for the risk-neutral measure $\mathbb{Q}$. The augmented market is also free of arbitrage since

$$\begin{aligned}\frac{\xi_t}{B_t} &= \frac{P(t,T)}{B_t}\ \mathbb{E}^{\mathbb{Q}_T}(\xi_T|\mathcal{F}_t) \\ &= \frac{P(t,T)}{B_t}\frac{\mathbb{E}^{\mathbb{Q}}(\xi e^{-\int_0^T r_s\ ds}|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s\ ds}|\mathcal{F}_t)} \\ &= \mathbb{E}^{\mathbb{Q}}(\xi_T e^{-\int_0^T r_s}|\mathcal{F}_t)\end{aligned}$$

and $(\xi_t/B_t)_{t\in[0,T]}$ is a martingale for $\mathbb{Q}$.]

**Problem 3.** (Vasicek model) Let

$$dr_t = -\lambda(\bar{r} - r_t)\ dt + \sigma\ dW_t$$

for positive constants $\lambda, \bar{r}$, and $\sigma$. Show that

$$\int_0^t r_s\ ds \sim N\left(\bar{r}t + (r_0 - \bar{r})\frac{(1-e^{-\lambda t})}{\lambda}, \frac{\sigma^2}{\lambda^2}\int_0^t (1-e^{-\lambda s})^2\ ds\right).$$

Use the moment generating function of a normal random variable to compute

$$f(0,T) = e^{-\lambda T}r_0 + (1-e^{-\lambda T})\bar{r} - \frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda T})^2$$

Show that if $r_0 \geq \bar{r}$ then $T \mapsto f(0,T)$ is decreasing.

*Solution* 3. The SDE of the Vasicek model can be solved explicitly:

$$r_t = r_0 e^{-\lambda t} + (1 - e^{-\lambda t})\bar{r} + \int_0^t e^{-\lambda(t-s)}\sigma dW_s.$$

Obviously, we have

$$\int_0^t \big(r_0 e^{-\lambda s} + (1 - e^{-\lambda s})\bar{r}\big) ds = \bar{r}t + (r_0 - \bar{r})\frac{1 - e^{-\lambda t}}{\lambda}.$$

Perhaps less obviously we have

$$\begin{aligned} \int_0^t \int_0^s e^{-\lambda(s-u)}\sigma dW_u ds &= \int_0^t \int_u^t e^{-\lambda(s-u)}\sigma ds dW_u \\ &= \int_0^t \frac{1 - e^{-\lambda(t-u)}}{\lambda}\sigma dW_u. \end{aligned}$$

As we have seen in Problem 3.3, since the integrand $\frac{1-e^{-\lambda(t-u)}}{\lambda}\sigma$ is not random, the stochastic integral is Gaussian with mean zero and variance given by Itô's isometry.

To shorten the notation, let

$$\begin{aligned} \mu(t) &= \bar{r}t + (r_0 - \bar{r})\frac{(1 - e^{-\lambda t})}{\lambda} \\ \sigma(t) &= \frac{\sigma}{\lambda}\left(\int_0^T (1 - e^{-\lambda s})^2 \, ds\right)^{1/2}. \end{aligned}$$

Using the well-known formula for the expectation of a log-normal distribution, we get the price at time 0 of the bond of maturity $T$:

$$P(0,T) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds}) = e^{-\mu(T) + \sigma(T)^2/2}$$

hence

$$\begin{aligned} f(0,T) &= -\frac{\partial \log P_0(T)}{\partial T} = \frac{\partial\big(\mu(T) - \frac{1}{2}\sigma(T)^2\big)}{\partial T} \\ &= \bar{r} + (r_0 - \bar{r})e^{-\lambda T} - \frac{\sigma^2}{2\lambda^2}\big(1 - e^{-\lambda T}\big)^2 \end{aligned}$$

Now, if $r_0 \geq \bar{r}$ we have the inequality

$$\frac{\partial f(0,T)}{\partial T} = -\lambda e^{-\lambda T}\Big((r_0 - \bar{r}) + \frac{\sigma^2}{\lambda^2}(1 - e^{-\lambda T})\Big) \leq 0$$

and hence the forward curve $T \to f(0,T)$ is decreasing.

**Problem 4.** (Vasicek model) Fix a time horizon $S > 0$. Express the dynamics of the short rate in the Vasicek model in terms of the $\mathbb{Q}_S$-Brownian motion, where $\mathbb{Q}_S$ is the $S$-forward measure. Hence, deduce that the distribution of $r_S$ under $\mathbb{Q}_S$ is

$$N\left(r_0 e^{-\lambda S} + (1 - e^{-\lambda S})\bar{r} - \frac{\sigma^2}{2\lambda^2}(1 - e^{-\lambda S})^2, \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda S})\right).$$

Use the Vasicek model to compute the time-0 price of a European call option maturing at time $S$ with strike $K$, written on a zero-coupon bond with maturity $T > S$.

*Solution* 4. Suppose the $\mathbb{Q}$ dynamics of the short rate $r$ are given by

$$dr_t = \lambda(\bar{r} - r_t)dt + \sigma dW_t$$

for a $\mathbb{Q}$-Brownian motion $W$.

Assume, for the moment, that there exists some process $\alpha$ such that

$$dW_t^{(S)} = dW_t + \alpha_t dt$$

defines a Brownian motion under $\mathbb{Q}_S$. We need only to find $\alpha$. But remember that the forward rate $(f(t,S))_{t\in[0,S]}$ is a martingale under $\mathbb{Q}_S$. Also, for the Vasicek model we have the formula

$$f(t,S) = e^{-\lambda(S-t)}r_t + (1 - e^{-\lambda(S-t)})\bar{r} - \frac{\sigma^2}{2\lambda^2}(1 - e^{-\lambda(S-t)})^2.$$

Hence by Itô's formula, we have

$$\begin{aligned} df(t,S) &= \lambda e^{-\lambda(S-t)}\left[r_t - \bar{r} + \frac{\sigma}{\lambda^2}(1 - e^{-\lambda(S-t)})\right]dt + e^{-\lambda(S-t)}dr_t \\ &= e^{-\lambda(S-t)}\left[dW_t + \frac{\sigma}{\lambda}(1 - e^{-\lambda(S-t)})dt\right] \end{aligned}$$

Hence $\alpha_t = \frac{\sigma}{\lambda}(1 - e^{-\lambda(S-t)})$ and the $\mathbb{Q}_S$ dynamics are given by

$$dr_t = \left(\lambda(\bar{r} - r_t) - \frac{\sigma^2}{\lambda}(1 - e^{-\lambda(S-t)})\right)dt + \sigma dW_t^{(S)}$$

It remains to prove the claim. Define the change of measure martingale $Z$ by

$$\begin{aligned} Z_t &= \mathbb{E}^{\mathbb{Q}}\left(\frac{d\mathbb{Q}_S}{d\mathbb{Q}}|\mathcal{F}_t\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left(\frac{e^{-\int_0^S r_s\ ds}}{P_0(S)}|\mathcal{F}_t\right) \\ &= \frac{e^{-\int_0^t r_s\ ds}P_t(S)}{P_0(S)}. \end{aligned}$$

Since $Z$ is adapted to the filtration generated by the Brownian motion, the martingale representation theorem says that

$$Z_t = 1 + \int_0^t k_s dW_s$$

for some process $k$. But since $Z$ is positive, we may let $\alpha_s = -k_t/Z_t$. Hence

$$Z_t = e^{-\frac{1}{2}\int_0^t \alpha_s^2 ds - \int_0^t \alpha_s dW_s}.$$

But by the Cameron–Martin–Girsanov theorem, the process

$$dW_t^{(S)} = dW_t + \alpha_t dt$$

defines a Brownian motion under $\mathbb{Q}_S$, as claimed.

[Here's another approach: We have seen in the Vasicek model the formula

$$\int_0^t r_s ds = r_0\frac{(1 - e^{-\lambda t})}{\lambda} + \int_0^t (1 - e^{-\lambda u})\bar{r}du + \int_0^t \frac{1 - e^{-\lambda(t-u)}}{\lambda}\sigma dW_u.$$

We have also seen

$$P(t,S)=\exp\left(-r_t\frac{(1-e^{-\lambda(S-t)})}{\lambda}-\bar r\int_0^{S-t}(1-e^{-\lambda u})\,du+\frac{\sigma^2}{2\lambda^2}\int_0^{S-t}(1-e^{-\lambda u})^2\,du\right).$$

so that when all is said and done:

$$Z_t=\frac{e^{-\int_0^t r_s\ ds}P(t,S)}{P(0,S)}=e^{-\frac{1}{2}\int_0^t\alpha_s^2ds-\int_0^t\alpha_s dW_s}$$

where $\alpha_t=\frac{\sigma}{\lambda}(1-e^{-\lambda(S-t)})$ so by the Cameron–Martin–Girsanov theorem, the process

$$dW_t^{(S)}=dW_t+\frac{\sigma}{\lambda}(1-e^{-\lambda(S-t)})dt$$

defines a Brownian motion under $\mathbb{Q}_S$. ]

Solving the Vasicek SDE yields

$$r_S=e^{-\lambda S}r_0+(1-e^{-\lambda S})\bar r+\int_0^S e^{-\lambda(S-u)}\sigma\left[dW_t^{(S)}-\frac{\sigma}{\lambda}(1-e^{-\lambda(S-u)})du\right]$$

and the distribution of $r_S$ under $\mathbb{Q}_S$ is verified.

Now, from Problem 2, we know the no-arbitrage price of the call is given by $P(0,S)\,\mathbb{E}^{\mathbb{Q}_S}[(P_S(T)-K)^+|\mathcal{F}_t]$. Also from Problem 2, we have the equality $\mathbb{E}^{\mathbb{Q}_S}[P(S,T)]=P(0,T)/P(0,S)$. Now, for the Vasicek model, $P(S,T)$ is log-normal under $\mathbb{Q}_S$ since

$$P_S(T)=\exp\left(-r_S\frac{(1-e^{-\lambda(T-S)})}{\lambda}-B(S,T)\right).$$

where $B(S,T)$ is not random. Let $v=\mathrm{Var}^{\mathbb{Q}_S}[\log P(S,T)]=\left(\frac{1-e^{-\lambda(T-S)}}{\lambda}\right)^2\frac{\sigma^2(1-e^{-2\lambda S})}{2\lambda}$. Hence, the price is given by

$$\begin{aligned}&P(0,S)\ \mathbb{E}[(e^{-v/2+\sqrt{v}Z}P_0(T)/P_0(S)-K)^+]\text{ where }Z\sim N(0,1)\\&=P(0,T)\ \mathrm{BS}\left(\frac{\sigma^2}{2\lambda^3}(1-e^{-\lambda(T-S)})^2(1-e^{-2\lambda S}),\frac{KP(0,S)}{P(0,T)}\right)\end{aligned}$$

where the function $\mathrm{BS}(v,m)=\Phi(-\log m/\sqrt{v}+\sqrt{v}/2)-m\Phi(-\log m/\sqrt{v}-\sqrt{v}/2)$ is the Black–Scholes call pricing function. Jamshidian found this formula in 1998.

[We could have computed the expectation $\mathbb{E}^{\mathbb{Q}}[e^{-\int_0^S r_s ds}(P_S(T)-K)^+]$ directly under the risk-neutral measure $\mathbb{Q}$, rather than switching to the forward measure $\mathbb{Q}_S$.

To do so, we notice that $X=\log P(S,T)$ and $Y=-\int_0^S r_u du$ are jointly normal random variables under $\mathbb{Q}$. Now we make use of the following formula:

$$\mathbb{E}[g(X)e^Y]=e^{\mu_Y+\sigma_Y^2/2}\mathbb{E}[g(X+\sigma_{X,Y})]$$

where $\mu_Y=\mathbb{E}(Y),\sigma_Y^2=\mathrm{Var}(Y)$, and $\sigma_{X,Y}=\mathrm{Cov}(X,Y)$. This formula can be proven by writing out the integrals in terms of the joint density function. Once we have this formula, we can proceeds along the lines of Problem 3.1. ]

**Problem 5.** (Hull–White extension of Cox–Ingersoll–Ross) Consider the short rate model given by

$$dr_t=-\lambda(\bar r(t)-r_t)\ dt+\sigma\sqrt{r_t}\ dW_t$$

for positive constants $\lambda$ and $\sigma$ and a deterministic function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}$. Find the initial forward rate curve $f_0 = f(0, \cdot)$ for this model. [Is it possible to find a $\bar{r}$ that exactly match a given $f_0$?]

*Solution* 5. Consider the PDE

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,T,r) + \lambda(\bar{r}(t) - r)\frac{\partial V}{\partial r}(t,T,r) + \frac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2}(t,T,r) = rV(t,T,r)\\ &V(T,T,r) = 1.\end{aligned}$$

As usual we can make the ansatz

$$V(t,T,r) = e^{-rA(t,T)-B(t,T)}$$

for some functions $A$ and $B$ which satisfy the boundary conditions $A(T,T) = B(T,T) = 0$. Substituting this into the PDE yields

$$-\frac{\partial A}{\partial t}(t,T)r - \frac{\partial B}{\partial t}(t,T))V(t,r) - \lambda(\bar{r}(t) - r)A(t,T)V(t,r) + \frac{\sigma^2}{2}rA(t,T)^2V(t,r) = rV(t,r).$$

which yields the coupled system

$$\begin{aligned}\frac{\partial A}{\partial t}(t,T) &= \lambda A(t) + \frac{\sigma^2}{2}A(t)^2 - 1\\ \frac{\partial B}{\partial t}(t,T) &= -\lambda\bar{r}(t)A(t,T).\end{aligned}$$

The equation for $A$ is a Riccati equation, whose solution is

$$\begin{aligned}A(t,T) &= \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma+\lambda)e^{\gamma(T-t)} + (\gamma - \lambda)}\\ B(t,T) &= \int_t^T \lambda\bar{r}(s)A(s,T)ds\end{aligned}$$

where $\gamma = \sqrt{\lambda^2 + 2\sigma^2}$. Hence, the time 0 forward rates are given by

$$f_0(T) = \frac{4\gamma^2 e^{\gamma T}}{[(\gamma+\lambda)e^{\gamma T} + (\gamma-\lambda)]^2}r_0 + \int_0^T \frac{4\gamma^2 e^{\gamma(T-s)}\lambda\bar{r}(s)ds}{[(\gamma+\lambda)e^{\gamma(T-s)} + (\gamma-\lambda)]^2}.$$

[Can can solve for $\bar{r}$ in terms of $f_0$? It seems pretty reasonable that this is possible, and indeed, we could certainly do it numerically, say, by discretizing time and solving a system of linear equations.

But is there an analytic solution? Notice that the above equations are of the convolution form

$$\int_0^T g(T-s)\bar{r}(s)ds = h(T)$$

If we define the Laplace transforms

$$G(x) = \int_0^\infty e^{-tx}g(t)dt,\ \bar{R}(x) = \int_0^\infty e^{-tx}\bar{r}(t)dt,\ \text{and}\ H(x) = \int_0^\infty e^{-tx}h(t)dt,$$

then the equation becomes $G(x)\bar{R}(x) = H(x) \Rightarrow \bar{R}(x) = H(x)/G(x)$. So we have an analytic solution as long as we can invert the Laplace transform.]

**Problem 6.** (interest rate swap) Party A agrees to give Party B a stream of payments throughout the interval $t \in [0, T]$, such that during the infinitesimal interval $(t, t+dt)$ Party B receives $r_t\, dt$ units of money, where $r_t$ is the short rate at time $t$. During the same period, Party B agrees to pay Party A $s\, dt$ units of money during the interval $(t, t+dt)$, where $s$ is a fixed constant. If no money changes hands at time 0, to what value should the constant $s$ be set so that there is no arbitrage? Your answer should be in terms of the time 0 bond prices $P(0, T)$. The constant $s = s(0, T)$ is called the swap rate for maturity $T$. Show that the left end point of the swap rate curve $s(0, \cdot)$ is the short rate $r_0$.

*Solution* 6. We assume the existence of an equivalent measure $\mathbb{Q}$ under which the prices of all assets discounted by the bank account are martingales. If the coupon payments of $r_{t_{i-1}}(t_i - t_{i-1})$ are made at the dates $t_0 \le t_1 \le ... \le t_{n-1} \le t_n$ then the time-0 price of this cash stream is

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=1}^{n} r_{t_{i-1}}(t_i - t_{i-1})e^{-\int_0^{t_i} r_u du}\Big) &\approx \mathbb{E}^{\mathbb{Q}}\left(\int_0^T r_t e^{-\int_0^t r_u du} dt\right)\\ &= \mathbb{E}^{\mathbb{Q}}(1 - e^{-\int_0^T r_u du})\\ &= 1 - P(0, T).\end{aligned}$$

Similarly, time-0 price of the stream of fixed payments $p\ (t_i - t_{i-1})$ is given by

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=1}^{n} s(t_i - t_{i-1})e^{-\int_0^{t_i} r_u du}\Big) &\approx \mathbb{E}^{\mathbb{Q}}\left(\int_0^T p e^{-\int_0^t r_u du} dt\right)\\ &= p\int_0^T P(0, u) du.\end{aligned}$$

The swap rate is given by

$$s(0, T) = \frac{1 - P(0, T)}{\int_0^T P(0, u) du}.$$

Note that

$$\lim_{T \downarrow 0} s(0, T) = -\frac{\partial}{\partial T} P(0, T)|_{T=0} = r_0.$$

**Problem 7.** Show that there is no arbitrage if the dynamics of the forward rates are given by

$$df(t, T) = \cos(T - t)[\sin(T - t)dt + dW_t]$$

for a scalar Brownian motion $(W_t)_{t \in \mathbb{R}_+}$.

*Solution* 7. This is an example of the HJM drift condition. If the volatility is given by $\sigma(t, T) = \cos(T - t)$, then there is no arbitrage if the drift is $\sigma(t, T)\int_t^T \sigma(t, s) ds = \cos(T - t)\sin(T - t)$. But to go through the HJM argument:

Fix $T > 0$ and observe

$$\begin{aligned} d\left(\int_0^t r_s \, ds + \int_t^T f(t,s) \, ds\right) &= (r_t - f(t,t))dt + \int_t^T df(t,s) \, ds \\ &= \int_t^T \cos(T-s)\sin(T-s)ds \; dt + \int_t^T \cos(T-s)dW_t \\ &= \frac{1}{2}\sin(T-t)^2 dt + \sin(T-t)dW_t \end{aligned}$$

since $f(t,t) = r_t$ and hence

$$d\tilde{P}(t,T) = de^{-\int_0^t r_s ds - \int_t^T f(t,s)ds} = -\tilde{P}(t,T)\sin(T-t)dW_t$$

by Itô's formula. Since $(\tilde{P}(t,T))_{t\in[0,T]}$ is a local martingale (in fact, a true martingale) for each $T > 0$, there is no arbitrage in the market.

**Problem 8.** Let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{E}(X_n) = 1$ for all $n$. Use the Borel–Cantelli lemma to show

$$\limsup_{n\to\infty} X_n^{1/n} \le 1 \text{ a.s.}$$

If $y(t,T)$ is the yield of a bond maturing at time $T$, the *long rate* is defined as $\ell_t = \lim_{T\to\infty} y(t,T)$ whenever the limit exists. Assuming that the bonds are priced by expectation, show that the long rate is non-decreasing, that is

$$\ell_t \ge \ell_s \text{ a.s. for all } 0 \le s \le t,$$

a fact first discovered by Dybvig, Ingersoll, and Ross in 1996.

*Solution* 8. For each $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n > (1+\epsilon)^n] \le \sum_{n=1}^{\infty} (1+\epsilon)^{-n} = \frac{1}{\epsilon} < \infty$$

by Markov's inequality. The first Borel–Cantelli lemma then says

$$\mathbb{P}(X_n^{1/n} > 1+\epsilon \text{ infinitely often}) = 0$$

This shows the event $\Omega_\epsilon = \{\limsup_{n\to\infty} X_n^{1/n} \le 1+\epsilon\}$ is almost sure for each $\epsilon > 0$. The first conclusion follows from the continuity of probability measures $\mathbb{P}(\cap_{k=1}^{\infty}\Omega_{1/k}) = 1$.

Now, let $P(t,T)$ be the bond price, $B_t$ the price of the money market, and $\tilde{P}(t,T) = P(t,T)/B_t$ the discounted bond price as usual. Suppose the long rate $\ell_t$ exists, so that

$$\ell_t = -\lim_{T\to\infty} \frac{1}{T}\log P(t,T) = -\lim_{T\to\infty}\frac{1}{T}\log \tilde{P}(t,T).$$

By assumption, the discounted bond prices are given by

$$\tilde{P}(t,T) = \mathbb{E}^{\mathbb{Q}}(B_T^{-1}|\mathcal{F}_t)$$

each $0 \le t \le T$ and a fixed risk-neutral measure $\mathbb{Q}$, and, in particular, $\tilde{P}(\cdot,T)$ is a martingale for each $T > 0$.

Fix $0 \le s \le t$, and let

$$X_n = \frac{\tilde{P}(t,n)}{\tilde{P}(s,n)}.$$

Note $\mathbb{E}(X_n) = \mathbb{E}[\mathbb{E}(X_n|\mathcal{F}_s)] = 1$ for each $n$. The first part implies

$$\ell_s - \ell_t = \lim_{n\to\infty} \frac{1}{n} \log X_n \leq 0 \text{ a.s.}$$

as required.