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Example sheet 4 - Michaelmas 2019

Problem 1. * Consider a three asset market with prices given by

$$\begin{aligned}\frac{dB_t}{B_t} &= 2\ dt\\ \frac{dS_t^{(1)}}{S_t^{(1)}} &= 3\ dt + dW_t^{(1)} - 2\ dW_t^{(2)}\\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= 5\ dt - 2\ dW_t^{(1)} + 4\ dW_t^{(2)}.\end{aligned}$$

Construct an absolute arbitrage.

Problem 2. (Black–Scholes formula) Let $X \sim N(0,1)$ be a standard normal random variable, and v and m be positive constants. Express the expectation

$$F(v,m) = \mathbb{E}[(e^{-v/2+\sqrt{v}X} - m)^+]$$

in terms of Φ, the distribution function of X. Prove the identity

$$F(v,m) = 1 - \frac{m^{1-p}e^{p(p-1)v/2}}{\sqrt{2\pi/v}}\mathbb{E}\left[\frac{e^{\mathrm{i}X(p-1/2-\log m)/\sqrt{v}}}{(X - \mathrm{i}p\sqrt{v})(X+\mathrm{i}(1-p)\sqrt{v})}\right]$$

holds for all $0 < p < 1$ and $v, m > 0$

Problem 3. (strictly local martingale in finance) Consider a market with zero interest rate $r = 0$ and stock price with dynamics

$$dS_t = S_t^2 dW_t.$$

Consider a European claim with payout $\xi_T = S_T$.
(a) Show that there exists a trading strategy which replicates the claim with corresponding wealth $\xi_t = V(t, S_t)$ where

$$V(t,S) = S\left[2\Phi\left(\frac{1}{S\sqrt{T-t}}\right) - 1\right].$$

(b) Consider the strategy of buying S_0 claims and selling ξ_0 shares. The time 0 wealth is $V_0 = 0$ and the time T wealth is $V_T = (S_0 - \xi_0)S_T > 0$. Is this strategy an absolute arbitrage?

Problem 4. (variance swap) Consider a market a stock with price S, where S be a positive Itô process, and interest rate $r = 0$. A variance swap is a European contingent claim with payout

$$\sum_{n=1}^{N}\left(\log\frac{S_{t_n}}{S_{t_{n-1}}}\right)^2.$$

where $0 \le t_0 < \cdots < t_N = T$ are fixed non-random dates. We know from stochastic calculus that the payout of the variance swap, for large N, is approximately given by

$$\xi_T = \langle \log S\rangle_T.$$

The goal of this exercise is to show that ξ_T can be replicated in an asymptotic sense.

(a) Confirm the identity

$$\log(S_T/S_0) = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2}\langle \log S \rangle_T.$$

(b) Confirm the identity

$$\log x = x - 1 - \int_0^1 \frac{(k-x)^+}{k^2} dk - \int_1^\infty \frac{(x-k)^+}{k^2} dk.$$

(c) Explain how to approximately replicate ξ_T by trading in the stock, cash, and a family of call and put options of different strikes but all with maturity date T. Show that the number of shares of the stock varies dynamically but the portfolio of calls and puts is static.

Problem 5. Consider a market with zero interest rate $r = 0$ and a stock with price dynamics

$$dS_t = S_t \sigma_t dW_t$$

where σ is independent of the $\mathbb{Q}$-Brownian motion W. Let

$$C(T, K) = \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+].$$

(a) Show that there is a family of measures μ_T on $[0, \infty)$ such that

$$C(T, K) = S_0 \int F(v, K/S_0) \mu_T(dv)$$

where F is the function defined in Problem 2.
(b) If there are constants $a \le b$ such that $a \le \sigma_t \le b$ a.s., show that the implied volatility satisfies

$$a \le \Sigma(T, K) \le b.$$

where the implied volatility $\Sigma(T, K)$ is defined implicitly as the unique non-negative solution σ of the equation

$$F(T\sigma^2, K/S_0) = C(T, K)/S_0.$$

(c) Show the equality $\Sigma(T, K) = \Sigma(T, S_0^2/K)$, i.e. the function $x \mapsto \Sigma(T, S_0 e^x)$ is even. Hint: First prove the identity

$$F(v, m) = 1 - m + mF(v, 1/m).$$

Problem 6. * (Hull–White extension of Cox–Ingersoll–Ross) Consider the short rate model given by

$$dr_t = \lambda(\bar{r}(t) - r_t)\, dt + \gamma \sqrt{r_t}\, dW_t$$

for positive constants λ and γ and a deterministic function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}$. Find the initial forward rate curve $T \mapsto f_0^T$ for this model.

Problem 7. Let $W^1, \ldots, W^m$ be m independent Brownian motions, and let $X^1, \ldots, X^m$ evolve as

$$dX_t^i = aX_t^i\, dt + b\, dW_t^i$$

given initial conditions $X_0^1, \ldots, X_0^m$ and fixed constants a, b.

Let

$$Z_t = \sum_{i=1}^m (X_t^i)^2 = \|X_t\|^2.$$

Show that there exists constants α, β, γ and a scalar Brownian motion $\hat{W}$ such that

$$dZ_t = (\alpha Z_t + \beta)dt + \gamma\sqrt{Z_t}d\hat{W}_t.$$

Problem 8. Given positive constants $\lambda, \bar{r}, \gamma$ such that $\bar{r} < \gamma$ and $\lambda\bar{r} > \gamma^2/2$, and an initial condition $0 < r_0 < \gamma$ and a Brownian motion W, it is possible to show that there exists a process $(r_t)_{t\geq 0}$ satisfying

$$dr_t = \lambda(\bar{r} - r_t)dt + (\gamma - r_t)\sqrt{r_t}dW_t$$

such that $0 < r_t < \gamma$ for all $t \geq 0$ almost surely. Define the function $H : \mathbb{R}_+ \times (0, \gamma) \to (0, 1)$ by

$$\mathbb{E}[e^{-\int_0^t r_s \, ds} | r_0 = r] = H(t, r)$$

Show that $H(t, \cdot)$ is a quadratic function for each $t \geq 0$.