Problem 1. * Consider a three asset market with prices given by

\[ \frac{dB_t}{B_t} = 2 \, dt \]

\[ \frac{dS^{(1)}_t}{S^{(1)}_t} = 3 \, dt + dW^{(1)}_t - 2 \, dW^{(2)}_t \]

\[ \frac{dS^{(2)}_t}{S^{(2)}_t} = 5 \, dt - 2 \, dW^{(1)}_t + 4 \, dW^{(2)}_t. \]

Construct an absolute arbitrage.

Problem 2. (Black–Scholes formula) Let \( X \sim N(0, 1) \) be a standard normal random variable, and \( v \) and \( m \) be positive constants. Express the expectation \( F(v, m) = E[(e^{-v/2 + \sqrt{v}X} - m)^+] \) in terms of \( \Phi \), the distribution function of \( X \). Prove the identity

\[ F(v, m) = 1 - \frac{m}{1 - p} e^{(p - 1/2 - \log m)/\sqrt{v}} \]

holds for all \( 0 < p < 1 \) and \( v, m > 0 \).

Problem 3. (strictly local martingale in finance) Consider a market with zero interest rate \( r = 0 \) and stock price with dynamics

\[ dS_t = S_t^2 dW_t. \]

(a) Show that there exists a trading strategy which replicates the claim with corresponding wealth \( \xi_t = V(t, S_t) \) where

\[ V(t, S) = S \left[ 2\Phi \left( \frac{1}{S\sqrt{T - t}} \right) - 1 \right]. \]

(b) Consider the strategy of buying \( S_0 \) claims and selling \( \xi_0 \) shares. The time 0 wealth is \( V_0 = 0 \) and the time \( T \) wealth is \( V_T = (S_0 - \xi_0)S_T > 0 \). Is this strategy an absolute arbitrage?

Problem 4. (variance swap) Consider a market a stock with price \( S \), where \( S \) be a positive Itô process, and interest rate \( r = 0 \). A variance swap is a European contingent claim with payout

\[ \frac{\sum_{n=1}^{N} \left( \log \frac{S_{t_n}}{S_{t_{n-1}}} \right)^2}{N} . \]

where \( 0 \leq t_0 < \cdots < t_N = T \) are fixed non-random dates. We know from stochastic calculus that the payout of the variance swap, for large \( N \), is approximately given by

\[ \xi_T = \langle \log S \rangle_T. \]

The goal of this exercise is to show that \( \xi_T \) can be replicated in an asymptotic sense.
(a) Confirm the identity
\[ \log(S_T/S_0) = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2} \langle \log S \rangle_T. \]
(b) Confirm the identity
\[ \log x = x - 1 - \int_0^1 \frac{(k-x)^+}{k^2} dk - \int_1^\infty \frac{(x-k)^+}{k^2} dk. \]
(c) Explain how to approximately replicate \( \xi_T \) by trading in the stock, cash, and a family of call and put options of different strikes but all with maturity date \( T \). Show that the number of shares of the stock varies dynamically but the portfolio of calls and puts is static.

**Problem 5.** Consider a market with zero interest rate \( r = 0 \) and a stock with price dynamics
\[ dS_t = S_t \sigma_t dW_t \]
where \( \sigma \) is independent of the \( \mathbb{Q} \)-Brownian motion \( W \). Let
\[ C(T, K) = \mathbb{E}_Q[(S_T - K)^+]. \]

(a) Show that there is a family of measures \( \mu_T \) on \( [0, \infty) \) such that
\[ C(T, K) = S_0 \int F(v, K/S_0) \mu_T(dv) \]
where \( F \) is the function defined in Problem 2.

(b) If there are constants \( a \leq b \) such that \( a \leq \sigma_t \leq b \) a.s., show that the implied volatility satisfies
\[ a \leq \Sigma(T, K) \leq b. \]
where the implied volatility \( \Sigma(T, K) \) is defined implicitly as the unique non-negative solution \( \sigma \) of the equation
\[ F(T \sigma^2, K/S_0) = C(T, K)/S_0. \]

(c) Show the equality \( \Sigma(T, K) = \Sigma(T, S_0^2/K) \), i.e. the function \( x \mapsto \Sigma(T, S_0 e^x) \) is even. Hint: First prove the identity
\[ F(v, m) = 1 - m + mF(v, 1/m). \]

**Problem 6.** *(Hull–White extension of Cox–Ingersoll–Ross)* Consider the short rate model given by
\[ dr_t = \lambda(\bar{r}(t) - r_t) \, dt + \gamma \sqrt{r_t} \, dW_t \]
for positive constants \( \lambda \) and \( \gamma \) and a deterministic function \( \bar{r} : \mathbb{R}_+ \to \mathbb{R} \). Find the initial forward rate curve \( T \mapsto f^T_0 \) for this model.

**Problem 7.** Let \( W^1, \ldots, W^m \) be \( m \) independent Brownian motions, and let \( X^1, \ldots, X^m \) evolve as
\[ dX_t^i = aX_t^i \, dt + b \, dW_t^i \]
given initial conditions \( X_0^1, \ldots, X_0^m \) and fixed constants \( a, b \).

Let
\[ Z_t = \sum_{i=1}^m (X_t^i)^2 = \|X_t\|^2. \]
Show that there exists constants $\alpha, \beta, \gamma$ and a scalar Brownian motion $\hat{W}$ such that
\[ dZ_t = (\alpha Z_t + \beta)dt + \gamma \sqrt{Z_t} d\hat{W}_t. \]

**Problem 8.** Given positive constants $\lambda, \bar{r}, \gamma$ such that $\bar{r} < \gamma$ and $\lambda \bar{r} > \gamma^2/2$, and an initial condition $0 < r_0 < \gamma$ and a Brownian motion $W$, it is possible to show that there exists a process $(r_t)_{t \geq 0}$ satisfying
\[ dr_t = \lambda(\bar{r} - r_t)dt + (\gamma - r_t) \sqrt{r_t} dW_t \]
such that $0 < r_t < \gamma$ for all $t \geq 0$ almost surely. Define the function $H : \mathbb{R}_+ \times (0, \gamma) \to (0, 1)$ by
\[ \mathbb{E}[e^{-\int_0^t r_s \, ds} | r_0 = r] = H(t, r) \]
Show that $H(t, \cdot)$ is a quadratic function for each $t \geq 0$. 