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Example sheet 4 - Michaelmas 2010

Problem 1. (Black–Scholes formula) Let $X \sim N(0,1)$ be a standard normal random variable, and v and m be positive constants. Compute

$$F(v,m) = \mathbb{E}[(e^{-v/2+\sqrt{v}X} - m)^+]$$

in terms of the standard normal distribution function $\Phi(t) = \mathbb{P}(X \leq t)$.

Problem 2. Consider a Black–Scholes market with two assets with dynamics given by

$$\begin{aligned} dB_t &= B_t \, r \, dt \\ dS_t &= S_t(\mu \, dt + \sigma dW_t) \end{aligned}$$

Find an arbitrage-free price $(\xi_t)_{t\in[0,T]}$ and the replicating strategy $(\pi_t)_{t\in[0,T]}$ for a claim with payout

(1) $\xi_T = S_T^p$ for some $p \in \mathbb{R}$
(2) $\xi_T = (\log S_T)^2$
(3) $\xi_T = \int_0^T S_s ds$

Show that your answer to part (3) is unchanged if $(S_t)_{t\in\mathbb{R}_+}$ is only assumed to be such that the market is free of arbitrage.

Problem 3. (Hull–White extension of Black–Scholes) Consider a market with zero interest rate and with initial stock price S_0. Suppose that there exists a family of call options, all struck at $K = S_0$, with maturities $0 < T_1 < T_2 < \ldots$ and prices $C(T_1), C(T_2), \ldots$. Show that as long as

$$0 \leq C(T_i) \leq C(T_{i+1}) < S_0$$

that there exists a non-random function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ such that the model with prices

$$dS_t = S_t \sigma(t) d\hat{W}_t$$

correctly prices all of the call options.

Problem 4. Consider a three asset market with prices are given by

$$\begin{aligned} \frac{dB_t}{B_t} &= 2 \, dt \\ \frac{dS_t^{(1)}}{S_t^{(1)}} &= 3 \, dt + dW_t^{(1)} - 2 \, dW_t^{(2)} \\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= 5 \, dt - 2 \, dW_t^{(1)} + 4 \, dW_t^{(2)}. \end{aligned}$$

Construct an arbitrage strategy.

Problem 5. Consider a two asset market with prices given by

$$\begin{aligned} \frac{dB_t}{B_t} &= 2 \, dt \\ \frac{dS_t}{S_t} &= 6 \, dt + 5 \, dW_t \end{aligned}$$

Show that if the filtration is generated by the market prices $(B_t, S_t)_{t\in\mathbb{R}_+}$ then the market is complete.

Now suppose

$$\begin{aligned}\frac{dB_t}{B_t} &= 2\ dt \\ \frac{dS_t}{S_t} &= 6\ dt + 3\ dW_t + 4\ dW_t^\perp\end{aligned}$$

Show that the prices in this new model have the same distribution as those in the previous model. However, show that if the filtration is generated by the independent Brownian motions $(W_t, W_t^\perp)_{t\in\mathbb{R}_+}$ then there exists more than one equivalent martingale measure.

Problem 6. (strictly local martingale) Consider a market with zero interest rate $r_t = 0$ and stock price with dynamics

$$dS_t = S_t^2 dW_t.$$

Consider an option with payout $\xi_T = S_T$. Show that there is no arbitrage if the price of this option is given by $\xi_t = V(t, S_t)$ where

$$V(t,S) = S\left[2\Phi\left(\frac{1}{S\sqrt{T-t}}\right) - 1\right]$$

where Φ is the standard normal distribution function. Why is this counter-intuitive?

Problem 7. (implied volatility) Consider a market with zero interest rate $r_t = 0$ and non-negative stock price $(S_t)_{t\in\mathbb{R}_+}$. Suppose the call prices are given by

$$C(T,K) = \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+]$$

where $\mathbb{Q}$ is an equivalent martingale measure.

The implied volatility $\Sigma(T,K)$ is defined implicitly as the unique non-negative solution of the equation

$$\mathrm{BS}(T\Sigma(T,K)^2, K/S_0) = C_0(T,K)/S_0$$

where

$$\mathrm{BS}(v,m) = \Phi(-\log m/\sqrt{v} + \sqrt{v}/2) - m\Phi(-\log m/\sqrt{v} - \sqrt{v}/2).$$

Show for fixed $T > 0$ the following inequality (due to Roger Lee in 2003)

$$\limsup_{K\to\infty} \frac{\sqrt{T}\Sigma(T,K)}{\sqrt{\log K}} \le \sqrt{2}.$$

Problem 8. (more implied volatility) Consider a market with zero interest rate $r_t = 0$, and a stock with dynamics

$$dS_t = S_t \sigma_t dW_t$$

where σ is independent of W. Suppose that the call options in this market are price by the formula

$$C(T,K) = \mathbb{E}[(S_T - K)^+].$$

Show that there is a family of measures μ_T on $[0,\infty)$ such that

$$C(T,K) = S_0 \int \mathrm{BS}(v, K/S_0)\mu_T(dv).$$

If there are constants $a \leq b$ such that $a \leq \sigma_t \leq b$ a.s., show that the implied volatility satisfies

$$a \leq \Sigma(T, K) \leq b.$$

*Can you show the equality $\Sigma(T, K) = \Sigma(T, S_0^2/K)$?

Problem 9. (local volatility) Consider the call surface $\{C(T, K) : T, K > 0\}$ in the previous problem. Show that there is a deterministic function $\hat{\sigma} : \mathbb{R}_+ \times \mathbb{R}_+ \to [a, b]$ such that

$$C(T, K) = \mathbb{E}[(\hat{S}_T - K)^+]$$

where

$$d\hat{S}_t = \hat{S}_t \sigma(t, \hat{S}_t) d\hat{W}_t$$

and $\hat{S}_0 = S_0$.

Problem 10. (Computer exercise) Consider a market with constant interest rate $r = 0.02$ and stock price dynamics $dS_t = S_t(\mu\ dt + \sigma(S_t)\ dW_t$ with initial condition $S_0 = 4$. The drift is given by $\mu = 0.09$ and the volatility is given by the function $\sigma(S) = 0.4 + 0.1 \times \cos(S)$.

Find a no-arbitrage price for a call option maturing at time $T = 1$ with strike $K = 3.5$. There are at least two approaches: You can either (1) numerically solve the pricing PDE, or (2) generate independent realizations of the stock price and appeal to the law of large numbers.

Problem 11. (Computer exercise) Today is the first of December, and the price of stock ABC on the first of the month over the past year is given below

Jan	£39	Jul	56
Feb	41	Aug	63
Mar	45	Sep	67
Apr	48	Oct	64
May	51	Nov	66
Jun	53	Dec	61

and the spot interest rate is 3% per year. Assuming you believe the Black–Scholes model, how much should you charge for a European call option on ABC with strike £60 and maturity date the first of December next year? How many shares of ABC should you now buy to hedge your exposure?

Problem 12. (Computer exercise) Today, the stock XYZ trades for £43. You have just sold a put option on XYZ with strike £40 and maturity date exactly one year from today for £3. The spot interest rate is 4% per year. Assuming you believe the Black–Scholes model, how many shares of XYZ should you now sell to hedge your exposure?