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Example sheet 4 - Michaelmas 2010

Problem 1. (Black–Scholes formula) Let $X \sim N(0,1)$ be a standard normal random variable, and v and m be positive constants. Compute

$$F(v,m) = \mathbb{E}[(e^{-v/2+\sqrt{v}X} - m)^+]$$

in terms of the standard normal distribution function $\Phi(t) = \mathbb{P}(X \le t)$.

Solution 1.

$$\begin{aligned}
\mathbb{E}[(e^{-v/2+\sqrt{v}X} - m)^+] &= \int_{-\infty}^{\infty} (e^{-v/2+\sqrt{v}x} - m)^+ \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{\log m/\sqrt{v}+\sqrt{v}/2}^{\infty} (e^{-v/2+\sqrt{v}x} - m) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{\log m/\sqrt{v}+\sqrt{v}/2}^{\infty} \frac{e^{-v/2+\sqrt{v}x-x^2/2}}{\sqrt{2\pi}} dx - m \int_{\log m/\sqrt{v}+\sqrt{v}/2}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{-\infty}^{-\log m/\sqrt{v}+\sqrt{v}/2} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds - m \int_{-\infty}^{-\log m/\sqrt{v}-\sqrt{v}/2} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \\
&= \boxed{\Phi\left(-\frac{\log m}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - m\, \Phi\left(-\frac{\log m}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right)}
\end{aligned}$$

where

$$\Phi(t) = \mathbb{P}(X \le t) = \int_{-\infty}^{t} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

is the standard normal distribution function.

Problem 2. Consider a Black–Scholes market with two assets with dynamics given by

$$\begin{aligned}
dB_t &= B_t\, r\, dt \\
dS_t &= S_t(\mu\, dt + \sigma dW_t)
\end{aligned}$$

Find the price $(\xi_t)_{t\in[0,T]}$ and the replicating strategy $(\pi_t)_{t\in[0,T]}$ for a claim with payout

(1) $\xi = S_T^p$ for some $p \in \mathbb{R}$
(2) $\xi = (\log S_T)^2$
(3) $\xi = \int_0^T S_s ds$

Show that your answer to part (3) is unchanged if $(S_t)_{t\in\mathbb{R}_+}$ is only assumed to be such that the market is free of arbitrage.

Solution 2. Note that $S_T = S_t e^{(r-\sigma^2/2)(T-t)+\sigma(\hat{W}_T - \hat{W}_t)}$ where $\hat{W}_t = W_t + (\mu - r)t/\sigma$ defines a Brownian motion for the equivalent martingale measure $\mathbb{Q}$.

(1) $\xi_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} S_T^p | \mathcal{F}_t] = S_t^p e^{(p-1)(r+p\sigma^2/2)(T-t)}$ and $\pi_t = pS_t^{p-1} e^{(p-1)(r+p\sigma^2/2)(T-t)}$
(2) $\xi_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(\log S_T)^2 | \mathcal{F}_t] = e^{-r(T-t)}\{[\log S_t + (r-\sigma^2/2)(T-t)]^2 + \sigma^2(T-t)\}$ and $\pi_t = 2e^{-r(T-t)}[\log S_t + (r-\sigma^2/2)(T-t)]/S_t$
(3) $\xi_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} \int_0^T S_s ds | \mathcal{F}_t] = e^{-r(T-t)} \int_0^t S_s ds + S_t \frac{1-e^{-r(T-t)}}{r}$ and $\pi_t = \frac{1-e^{-r(T-t)}}{r}$.

Notice in (3) that the answer does not depend on σ. This is a hint that the price and hedge are *independent* of the model for $(S_t)_{t\in[0,T]}$. Indeed, note that

$$\begin{aligned}\tilde{\xi}_T &= e^{-rT}\int_0^T S_s ds \\ &= \int_0^T e^{-r(T-s)}\tilde{S}_s ds \\ &= S_0\frac{1-e^{-rT}}{r} + \int_0^T \frac{1-e^{-r(T-s)}}{r} d\tilde{S}_s\end{aligned}$$

and hence a replicating strategy is $\pi_t = \frac{1-e^{-r(T-t)}}{r}$.

Problem 3. (Hull–White extension of Black–Scholes) Consider a market with zero interest rate and with initial stock price S_0. Suppose that there exists a family of call options, all struck at $K = S_0$, with maturities $0 < T_1 < T_2 < \ldots$ and prices $C(T_1), C(T_2), \ldots$. Show that as long as

$$0 \leq C(T_i) \leq C(T_{i+1}) < S_0$$

that there exists a non-random function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ such that the model with prices

$$dS_t = S_t\sigma(t)d\hat{W}_t$$

correctly prices all of the call options.

Solution 3. There would be no arbitrage if

$$C(T) = \mathbb{E}[(S_T - S_0)^+]$$

for each maturity T. But, note that $d\log S_t = -\frac{1}{2}\sigma(t)^2dt + \sigma(t)dW_t$ by Itô's formula. Since σ is not random, we can conclude that

$$\log S_t \sim N\left(\log(S_0) - \frac{1}{2}\int_0^t \sigma(s)^2ds, \int_0^t \sigma(s)^2ds\right)$$

Hence, the local volatility model prices the call with maturity T as

$$C(T) = S_0\mathrm{BS}\left(\int_0^T \sigma(s)^2ds, 1\right) = S_0\left\{2\Phi\left[\frac{1}{2}\left(\int_0^t \sigma(s)^2ds\right)^{1/2}\right] - 1\right\}$$

Hence, we need only choose σ in such a way that

$$\int_0^{T_i} \sigma(s)^2ds = \left[2\Phi^{-1}\left(\frac{C(T_i)}{2S_0} + \frac{1}{2}\right)\right]^2$$

for each i. But this is always possible by the assumption the the call prices are increasing in maturity and bound by the initial stock price.

Problem 4. Consider a three asset market with prices are given by

$$\begin{aligned}\frac{dB_t}{B_t} &= 2\ dt\\ \frac{dS_t^{(1)}}{S_t^{(1)}} &= 3\ dt + dW_t^{(1)} - 2\ dW_t^{(2)}\\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= 5\ dt - 2\ dW_t^{(1)} + 4\ dW_t^{(2)}.\end{aligned}$$

Construct an arbitrage strategy.

Solution 4. Note that

$$\begin{aligned}\frac{d\tilde{S}_t^{(1)}}{\tilde{S}_t^{(1)}} &= \ dt + dW_t^{(1)} - 2\ dW_t^{(2)}\\ \frac{d\tilde{S}_t^{(2)}}{\tilde{S}_t^{(2)}} &= 3\ dt - 2\ dW_t^{(1)} + 4\ dW_t^{(2)}.\end{aligned}$$

Let

$$\pi_t^1 = \frac{2}{\tilde{S}_t^{(1)}} \text{ and } \pi_t^2 = \frac{1}{\tilde{S}_t^{(2)}}$$

Then

$$\int_0^T \pi_s \cdot d\tilde{S}_s = \int_0^T 2(dt + dW_t^{(1)} - 2\ dW_t^{(2)}) + (3\ dt - 2\ dW_t^{(1)} + 4\ dW_t^{(2)}) = 5T > 0.$$

Problem 5. Consider a two asset market with prices given by

$$\begin{aligned}\frac{dB_t}{B_t} &= 2\ dt\\ \frac{dS_t}{S_t} &= 6\ dt + 5\ dW_t\end{aligned}$$

Show that if the filtration is generated by the market prices $(B_t, S_t)_{t\in\mathbb{R}_+}$ then the market is complete.

Now suppose

$$\begin{aligned}\frac{dB_t}{B_t} &= 2\ dt\\ \frac{dS_t}{S_t} &= 6\ dt + 3\ dW_t + 4\ dW_t^{\perp}\end{aligned}$$

Show that the prices in this new model have the same distribution as those in the previous model. However, show that if the filtration is generated by the independent Brownian motions $(W_t, W_t^{\perp})_{t\in\mathbb{R}_+}$ then there exists more than one equivalent martingale measure.

Solution 5. Consider the first market. Let $\hat{W}_t = W_t + 4/5t$. We need only see that $\{S_t \leq x\} = \{\hat{W}_t \leq \frac{1}{5}\log(x/S_0) - \frac{2}{5}t\}$ for $x > 0$, so that the processes $(S_t)_{t\in\mathbb{R}_+}$ and $(\hat{W}_t)_{t\in\mathbb{R}_+}$ generate the same filtration. Hence, the market is complete.

Now, letting $Y_t = \frac{1}{5}(3W_t + 4W_t^{\perp})$, we can verify that $(Y_t)_{t\in\mathbb{R}_+}$ is a Brownian motion from the definition. Hence the two models have the same statistics. Now, we have seen that if the market is complete, then the martingale measure is unique. On the other hand, for every $a \in \mathbb{R}$, the following martingale

$$Z_t^{(a)} = e^{-\frac{4}{3}aW_t-(1-a)W_t^{\perp}-(1-2a+25a^2/9)T/2}$$

defines a measure $\mathbb{Q}^{(a)}$ for which the processes

$$\begin{aligned} \hat{W}_t &= W_t + \frac{4}{3}at \\ \hat{W}_t^{\perp} &= W_t^{\perp} + (1-a)t \end{aligned}$$

are Brownian motions. Hence, the discounted stock price

$$d\tilde{S}_t = \tilde{S}_t(4\ dt + 3\ dW_t + 4\ dW_t^{\perp}) = \tilde{S}_t(3\ d\hat{W}_t + 4\ d\hat{W}_t^{\perp})$$

is a martingale for each a.

Problem 6. (strictly local martingale) Consider a market with zero interest rate $r_t = 0$ and stock price with dynamics

$$dS_t = S_t^2 dW_t.$$

Consider an option with payout $\xi_T = S_T$. Show that there is no arbitrage if the price of this option is given by $\xi_t = V(t, S_t)$ where

$$V(t,S) = S\left[2\Phi\left(\frac{1}{S\sqrt{T-t}}\right) - 1\right]$$

where Φ is the standard normal distribution function. Why is this counter-intuitive?

Solution 6. It is straight-forward, if a bit tedious, to verify

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^4\frac{\partial^2 V}{\partial S^2} = 0$$

and $\lim_{t\uparrow T} V(t,S) = S$. Since $(S_t, \xi_t)_{t\in[0,T]}$ is a local martingale, there is no arbitrage. Furthermore, the claim with payout $\xi_T = S_T$ can be replicated

$$S_T = V(0,S_0) + \int_0^T \pi_s dS_s.$$

But since $V(0,S_0) < S_0$, the above equation says that one can replicate the stock at some time $T > 0$, by trading in the very same stock, in such a way that the replication cost is strictly less than the current price of the stock!

Problem 7. (implied volatility) Consider a market with zero interest rate $r_t = 0$ and non-negative stock price $(S_t)_{t\in\mathbb{R}_+}$. Suppose the call prices are given by

$$C(T,K) = \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+]$$

where $\mathbb{Q}$ is an equivalent martingale measure.

The implied volatility $\Sigma(T,K)$ is defined implicitly as the unique non-negative solution of the equation

$$\mathrm{BS}(T\Sigma(T,K)^2, K/S_0) = C(T,K)/S_0$$

where

$$\mathrm{BS}(v,m) = \Phi(-\log m/\sqrt{v} + \sqrt{v}/2) - m\Phi(-\log m/\sqrt{v} - \sqrt{v}/2).$$

Show for fixed $T > 0$ the following inequality (due to Roger Lee in 2003)

$$\limsup_{K\to\infty} \frac{\sqrt{T}\Sigma(T,K)}{\sqrt{\log K}} \le \sqrt{2}.$$

Solution 7. By the monotone convergence theorem, we have $\lim_{K\to\infty} C(T,K) = 0$ and hence

$$\lim_{K\to\infty} \mathrm{BS}(T\Sigma(T,K)^2, K) = 0 \tag{1}$$

On the other hand, we have

$$\lim_{K\to\infty} \mathrm{BS}(2\log K, K) = \Phi(0) - \lim_{K\to\infty} K\Phi\big(-\sqrt{2\log K}\big) = \frac{1}{2} \tag{2}$$

[We have used

$$\lim_{x\uparrow\infty} \Phi(-x)e^{x^2/2} = \lim_{x\uparrow\infty} \frac{\Phi(-x)}{e^{-x^2/2}} = \lim_{x\uparrow\infty} \frac{-\frac{1}{\sqrt{2\pi}}e^{-x^2/2}}{-xe^{-x^2/2}} = 0$$

by l'Hôpital's rule in the second term above.]

From (1) and (2) combined with the fact that the map $\sigma \mapsto \mathrm{BS}(\sigma^2 T, K)$ is strictly increasing, we conclude that there must exist a value K_0 such that:

$$\Sigma(T,K) < \sqrt{\frac{2\log K}{T}}, \quad \text{for all } K > K_0$$

a conclusion which is stronger than what we were asked to prove.

Problem 8. (more implied volatility) Consider a market with zero interest rate $r_t = 0$, and a stock with dynamics

$$dS_t = S_t\sigma_t dW_t$$

where σ is independent of W. Suppose that the call options in this market are price by the formula

$$C(T,K) = \mathbb{E}[(S_T - K)^+].$$

Show that there is a family of measures μ_T on $[0,\infty)$ such that

$$C(T,K) = S_0 \int \mathrm{BS}(v, K/S_0)\mu_T(dv).$$

If there are constants $a \le b$ such that $a \le \sigma_t \le b$ a.s., show that the implied volatility satisfies

$$a \le \Sigma(T,K) \le b.$$

*Can you show the equality $\Sigma(T,K) = \Sigma(T, S_0^2/K)$?

Solution 8. Notice that conditional on the process σ, the distribution of

$$\log S_T = \log S_0 - \frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s$$

is normal, since σ and W are independent. Hence

$$\mathbb{E}[(S_T - K)^+|\sigma] = S_0 \ \mathrm{BS}\left(\int_0^T \sigma_s^2 ds, K/S_0\right).$$

The conclusion follows from the tower property of conditional expectations, where the measure μ_T is the law of the non-negative random variable $\int_0^T \sigma_s^2 \, ds$.

Since $v \mapsto \mathrm{BS}(v,m)$ is increasing, we have

$$\mathrm{BS}(Ta^2, K/S_0) \le \int \mathrm{BS}(v, K/S_0)\mu_T(dv) \le \mathrm{BS}(Tb^2, K/S_0).$$

But since the middle term is just $\mathrm{BS}(T\Sigma(T,K)^2, K/S_0)$, we can conclude

$$a \le \Sigma(T,K) \le b.$$

* Notice the Black–Scholes call price function satisfies the following identity:

$$\begin{aligned} \mathrm{BS}(v,m) &= \Phi(-\log m/\sqrt{v} + \sqrt{v}/2) - m\Phi(-\log m/\sqrt{v} - \sqrt{v}/2) \\ &= 1 - \Phi(\log m/\sqrt{v} - \sqrt{v}/2) - m + \Phi(\log m/\sqrt{v} + \sqrt{v}/2) \\ &= 1 - m + m\ \mathrm{BS}(v, 1/m) \end{aligned}$$

Now use the above calculation:

$$\begin{aligned} S_0\ \mathrm{BS}(T\Sigma(T,K)^2, K/S_0) &= \int S_0\ \mathrm{BS}(v, K/S_0)\mu_T(dv) \\ &= \int [S_0 - K + K\ \mathrm{BS}(v, S_0/K)]\mu_T(dv) \\ &= S_0 - K + K\ \mathrm{BS}(T\Sigma(T, S_0^2/K), S_0/K) \\ &= S_0\ \mathrm{BS}(T\Sigma(T, S_0^2/K)^2, K/S_0). \end{aligned}$$

In particular, the implied volatility smile is symmetric as a function of $\log(K/S_0)$. This observation is due to Renault and Touzi in 1996.

Problem 9. (local volatility) Consider the call surface $\{C(T,K) : T, K > 0\}$ in the previous problem. Show that there is a deterministic function $\hat{\sigma} : \mathbb{R}_+ \times \mathbb{R}_+ \to [a,b]$ such that

$$C(T,K) = \mathbb{E}[(\hat{S}_T - K)^+]$$

where

$$d\hat{S}_t = \hat{S}_t \sigma(t, \hat{S}_t) d\hat{W}_t$$

and $\hat{S}_0 = S_0$.

Solution 9. By Dupire's theorem, the local volatility function is given by

$$\sigma(T,K)^2 = \frac{2\frac{\partial C(T,K)}{\partial T}}{K^2 \frac{\partial^2 C(T,K)}{\partial K^2}}.$$

where

$$C(T,K) = \mathbb{E}[(S_T - K)^+] = S_0 \mathbb{E}\left[\mathrm{BS}\left(\int_0^T \sigma_s^2 ds, K/S_0\right)\right]$$

Define a function G by

$$G(v,m) = \frac{1}{\sqrt{v}}\ \phi\big(-\log m/\sqrt{v} + \sqrt{v}/2\big)$$

where $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. After some calculations, we get

$$\frac{\partial}{\partial v}\mathrm{BS}(v,m) = \frac{1}{2}G(v,m)$$

and

$$\frac{\partial^2}{\partial m^2}\mathrm{BS}(v,m) = \frac{1}{m^2}G(v,m).$$

[Note, you have to use the identity $\phi(-\log m/\sqrt{v}+\sqrt{v}/2) = m\phi(-\log m/\sqrt{v}-\sqrt{v}/2)$.]

So that

$$\begin{aligned}\sigma(T,K)^2 &= \frac{2\mathbb{E}\left[\sigma_T^2 \frac{\partial}{\partial v}\mathrm{BS}\left(\int_0^T \sigma_s^2 ds, K/S_0\right)\right]}{K^2\mathbb{E}\left[\left(\frac{1}{S_0}\right)^2 \frac{\partial^2}{\partial m^2}\mathrm{BS}\left(\int_0^T \sigma_s^2 ds, K/S_0\right)\right]} \\ &= \frac{\mathbb{E}\left[\sigma_T^2 G\left(\int_0^T \sigma_s^2 ds, K/S_0\right)\right]}{\mathbb{E}\left[G\left(\int_0^T \sigma_s^2 ds, K/S_0\right)\right]}\end{aligned}$$

Since G is everywhere positive, we see that the squared local volatility $\sigma(T,K)^2$ is an average of the squared spot volatility σ_T^2 with respect to an equivalent probability measure $\mathbb{P}^{T,K}$ with density proportional to $G\left(\int_0^T \sigma_s^2 ds, K/S_0\right)$. In particular, we can conclude that $a^2 \leq \sigma(T,K)^2 \leq b^2$.

Problem 10. (Computer exercise) Consider a market with constant interest rate $r = 0.02$ and stock price dynamics $dS_t = S_t(\mu\, dt + \sigma(S_t)\, dW_t)$ with initial condition $S_0 = 4$. The drift is given by $\mu = 0.09$ and the volatility is given by the function $\sigma(S) = 0.4 + 0.1 \times \cos(S)$.

Find a no-arbitrage price for a call option maturing at time $T = 1$ with strike $K = 3.5$. There are at least two approaches: You can either (1) numerically solve the pricing PDE, or (2) generate independent realizations of the stock price and appeal to the law of large numbers.

Solution 10. The time-0 price (more accurately: the minimal replication cost = expectation under the unique equivalent martingale measure) is approximately 0.82 and the time-0 hedging portfolio is approximately 0.60 shares.

Problem 11. (Computer exercise) Today is the first of December, and the price of stock ABC on the first of the month over the past year is given below

Jan	£39	Jul	56
Feb	41	Aug	63
Mar	45	Sep	67
Apr	48	Oct	64
May	51	Nov	66
Jun	53	Dec	61

and the spot interest rate is 3% per year. Assuming you believe the Black–Scholes model, how much should you charge for a European call option on ABC with strike £60 and maturity date the first of December next year? How many shares of ABC should you now buy to hedge your exposure?

Solution 11. As this question involves statistics, it is open ended: there really is no right or wrong answer. Here is one possible analysis:

According to the Black–Scholes model, $\log S_t = \log S_0 + \nu t + \sigma W_t$ for constants ν and σ. We can estimate the unknown parameters by noting that the $n = 11$ increments $\log(S_{t_i}/S_{t_{i-1}})$

are independent and $N(\nu(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1}))$. The maximum likelihood estimator of the growth rate ν is

$$\hat{\nu} = \frac{1}{t_n - t_0} \sum_{i=1}^{N} \log(S_{t_i}/S_{t_{i-1}}) = 12/11 \times \log(61/39) \approx 0.212$$

where we've made the approximation $t_i - t_{i-1} = 1/12$. The maximum likelihood estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{[\log(S_{t_i}/S_{t_{i-1}}) - (t_i - t_{i-1})\hat{\nu}]^2}{t_i - t_{i-1}} \approx 0.00663$$

so that $\hat{\sigma} = 0.0815$. Using this volatility, the Black–Scholes price of the option is then

$$S_0 \mathrm{BS}(\sigma^2 T, \log(K/S_0) - rT) \approx 3.63.$$

The hedge is given by $\Phi\left(-\frac{\log(K/S_0)}{\sigma\sqrt{T}} + (r/\sigma + \sigma/2)\sqrt{T}\right) = 0.730$.

You may object to the choice of the maximum likelihood estimator $\hat{\sigma}^2$ since it is biased. You may prefer, then, to use the unbiased estimator

$$\tilde{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} \frac{[\log(S_{t_i}/S_{t_{i-1}}) - (t_i - t_{i-1})\hat{\nu}]^2}{t_i - t_{i-1}} \approx 0.00730$$

so $\tilde{\sigma} \approx 0.0854$. The Black–Scholes price is then about 3.71, and the hedge is about 0.722.

If you want to be even more refined, you could contruct a confidence interval for your estimate of σ. Since you are selling the call, you are probably worried about underpricing. Since the Black–Scholes call price is increasing in volatility, we need only find a good upper bound on σ. Notice that since $n\hat{\sigma}^2/\sigma^2$ has the chi-squared distribution with $n - 1 = 10$ degrees of freedom, we have

$$\mathbb{P}(n\hat{\sigma}^2/\sigma^2 \geq 3.94) \approx 0.95.$$

Therefore, we can say with about 95 % confidence that σ is smaller than $\hat{\sigma}\sqrt{11/3.94} \approx 0.136$. The corresponding Black–Scholes price is approximately 4.81 with hedge 0.659 shares.

Problem 12. (Computer exercise) Today, the stock XYZ trades for £43. You have just sold a put option on XYZ with strike £40 and maturity date exactly one year from today for £3. The spot interest rate is 4% per year. Assuming you believe the Black–Scholes model, how many shares of XYZ should you now sell to hedge your exposure?

Solution 12. The Black–Scholes put price is given by

$$Ke^{-rT}\Phi\left(\frac{\log(K/S_0)}{\sigma\sqrt{T}} - (r/\sigma - \sigma/2)\sqrt{T}\right) - S_0\Phi\left(\frac{\log(K/S_0)}{\sigma\sqrt{T}} - (r/\sigma + \sigma/2)\sqrt{T}\right)$$

which can be derived either by computing $\mathbb{E}^{\mathbb{Q}}[e^{-rT}(K - S_T)^+]$ or by applying put/call parity to the Black–Scholes call price. The implied volatility is can be found a root-finding algorithm to be about 0.306. According to the Black–Scholes model, the hedge is

$$-\Phi\left(\frac{\log(K/S_0)}{\sigma\sqrt{T}} - (r/\sigma + \sigma/2)\sqrt{T}\right) \approx -0.302.$$