Advanced Financial Models Michael Tehranchi

Example sheet 4 - Michaelmas 2019

Problem 1. * Consider a three asset market with prices given by

$$\begin{aligned}
\frac{dB_t}{B_t} &= 2\ dt \\
\frac{dS_t^{(1)}}{S_t^{(1)}} &= 3\ dt + dW_t^{(1)} - 2\ dW_t^{(2)} \\
\frac{dS_t^{(2)}}{S_t^{(2)}} &= 5\ dt - 2\ dW_t^{(1)} + 4\ dW_t^{(2)}.
\end{aligned}$$

Construct an absolute arbitrage.

Solution 1. If the pure investment strategy is decomposed as (ϕ, π) a good choice for the holding is stock is given by

$$\pi_t = \left(\frac{2}{S_t^{(1)}}, \frac{1}{S_t^{(2)}} \right)$$

but, of course, it is not unique. It remains to find the holding in the bank account ϕ. Note that the wealth X evolves as

$$\begin{aligned}
dX_t &= r(X_t - \pi_t \cdot S_t)dt + \pi_t \cdot dS_t \\
&= (2X_t + 5)dt
\end{aligned}$$

so the unique solution with $X_0 = 0$ is

$$X_t = \frac{5}{2}(e^{2t} - 1).$$

Since $X_t > 0$ a.s. for $t > 0$, this is an arbitrage with the holding in the bank account given by

$$\phi_t = \frac{X_t - \pi_t \cdot S_t}{B_t} = \frac{5 - 11e^{-2t}}{2B_0}.$$

Problem 2. (Black–Scholes formula) Let $X \sim N(0, 1)$ be a standard normal random variable, and v and m be positive constants. Express the expectation

$$F(v, m) = \mathbb{E}[(e^{-v/2+\sqrt{v}X} - m)^+]$$

in terms of Φ, the distribution function of X. Prove the identity

$$F(v, m) = 1 - \frac{m^{1-p} e^{p(p-1)v/2}}{\sqrt{2\pi/v}} \mathbb{E}\left[\frac{e^{\mathrm{i}X(p-1/2-\log m)/\sqrt{v}}}{(X - \mathrm{i}p\sqrt{v})(X + \mathrm{i}(1-p)\sqrt{v})} \right]$$

holds for all $0 < p < 1$ and $v, m > 0$

Solution 2.

$$\begin{aligned}
\mathbb{E}[(e^{-v/2+\sqrt{v}X}-m)^+] &= \int_{-\infty}^{\infty}(e^{-v/2+\sqrt{v}x}-m)^+\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \int_{\log m/\sqrt{v}+\sqrt{v}/2}^{\infty}(e^{-v/2+\sqrt{v}x}-m)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \int_{\log m/\sqrt{v}+\sqrt{v}/2}^{\infty}\frac{e^{-v/2+\sqrt{v}x-x^2/2}}{\sqrt{2\pi}}dx - m\int_{\log m/\sqrt{v}+\sqrt{v}/2}^{\infty}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \int_{-\infty}^{-\log m/\sqrt{v}+\sqrt{v}/2}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds - m\int_{-\infty}^{-\log m/\sqrt{v}-\sqrt{v}/2}\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt \\
&= \boxed{\Phi\left(-\frac{\log m}{\sqrt{v}}+\frac{\sqrt{v}}{2}\right) - m\,\Phi\left(-\frac{\log m}{\sqrt{v}}-\frac{\sqrt{v}}{2}\right)}
\end{aligned}$$

The second identity is consequence of the formula relating call prices to the moment generating function of a log stock price. To apply the formula, note that $M(\theta)=\mathbb{E}[(e^{-v/2+\sqrt{v}X})^\theta]=e^{\theta(\theta-1)v/2}$ hence

$$F(v,m)=1-\frac{m^{1-p}}{2\pi}\int\frac{e^{p(p-1)v/2+\mathrm{i}x(p-1/2)v-x^2v/2}e^{-\mathrm{i}x\log m}}{(x-\mathrm{i}p)(x+\mathrm{i}(1-p))}dx$$

Problem 3. (strictly local martingale in finance) Consider a market with zero interest rate $r=0$ and stock price with dynamics

$$dS_t = S_t^2 dW_t.$$

Consider a European claim with payout $\xi_T = S_T$.
(a) Show that there exists a trading strategy which replicates the claim with corresponding wealth $\xi_t = V(t,S_t)$ where

$$V(t,S)=S\left[2\Phi\left(\frac{1}{S\sqrt{T-t}}\right)-1\right].$$

(b) Consider the strategy of buying S_0 claims and selling ξ_0 shares. The time 0 wealth is $V_0=0$ and the time T wealth is $V_T=(S_0-\xi_0)S_T>0$. Is this strategy an absolute arbitrage?

Solution 3. (a) It is straight-forward, if a bit tedious, to verify

$$\frac{\partial V}{\partial t}+\frac{1}{2}S^4\frac{\partial^2 V}{\partial S^2}=0$$

and $\lim_{t\uparrow T}V(t,S)=S$. The replication strategy is given by $\pi_t=\frac{\partial V}{\partial S}(t,S_t)$ as usual.
(b) This candidate arbitrage is not admissible. Indeed, let $V_t=S_0\xi_t-\xi_0 S_t$. Note that V is a local martingale, as it is the linear combination of two local martingales. Now if the strategy were admissible, the wealth V would be bounded from below and hence a supermartingale. Therefore we would have to conclude

$$0=V_0\geq\mathbb{E}(V_T)=\mathbb{E}(S_0\xi_T-\xi_0 S_T)=(S_0-\xi_0)\mathbb{E}(S_T)$$

But this contradicts $S_0>\xi_0$. So in this market it is impossible to lock in the sure future profit at zero initial cost, because doing so leaves open the possibility that the wealth is goes negative between times $t=0$ and $t=T$.

Note, however, that there *is* an admissible arbitrage relative to the asset with price S. Indeed, the strategy of holding one share of the claim is a relative arbitrage, since the initial discounted wealth is $\xi_0/S_0 < 1$ and the terminal discounted wealth is $\xi_T/S_T = 1$. This is only a relative arbitrage since you do not short S, and hence must start with positive initial wealth.

Problem 4. (variance swap) Consider a market a stock with price S, where S be a positive Itô process, and interest rate $r = 0$. A variance swap is a European contingent claim with payout

$$\sum_{n=1}^{N} \left(\log \frac{S_{t_n}}{S_{t_{n-1}}} \right)^2.$$

where $0 \le t_0 < \cdots < t_N = T$ are fixed non-random dates. We know from stochastic calculus that the payout of the variance swap, for large N, is approximately given by

$$\xi_T = \langle \log S \rangle_T.$$

The goal of this exercise is to show that ξ_T can be replicated in an asymptotic sense.
(a) Confirm the identity

$$\log(S_T/S_0) = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2}\langle \log S \rangle_T.$$

(b) Confirm the identity

$$\log x = x - 1 - \int_0^1 \frac{(k-x)^+}{k^2} dk - \int_1^\infty \frac{(x-k)^+}{k^2} dk.$$

(c) Explain how to approximately replicate ξ_T by trading in the stock, cash, and a family of call and put options of different strikes but all with maturity date T. Show that the number of shares of the stock varies dynamically but the portfolio of calls and puts is static.

Solution 4. (a) Note that by Itô's formula:

$$d \log S_t = \frac{dS_t}{S_t} - \frac{d\langle S \rangle_t}{2S_t^2}.$$

and hence

$$d\langle \log S \rangle_t = \frac{d\langle S \rangle_t}{S_t^2}.$$

The conclusion follows.
(b) Since $(k-x)^+ \ge 0$ only if $k \ge x$ we have

$$\begin{aligned} \int_0^1 \frac{(k-x)^+}{k^2} dk &= \int_{x \wedge 1}^1 \left(\frac{1}{k} - \frac{x}{k^2} \right) dk \\ &= -\log(x \wedge 1) - (1-x)^+ \end{aligned}$$

Similarly,

$$\begin{aligned} \int_1^\infty \frac{(x-k)^+}{k^2} dk &= \int_1^{x \vee 1} \left(\frac{x}{k^2} - \frac{1}{k} \right) dk \\ &= (x-1)^+ - \log(x \vee 1) \end{aligned}$$

(c) By the construction of quadratic variation in lecture, we have

$$\begin{aligned}
\xi_T &\approx \langle \log S \rangle_T \\
&= 2\int_0^T \frac{dS_t}{S_t} - 2\log(S_T/S_0) \\
&= 2\int_0^T \frac{dS_t}{S_t} - 2S_T + 2(\log S_0 + 1) + 2\int_0^1 \frac{P_T(T,K)}{K^2}dK + 2\int_1^\infty \frac{C_T(T,K)}{K^2}dK
\end{aligned}$$

where $P_T(T,K) = (K - S_T)^+$ is the payout of a put option and $C_T(T,K) = (S_T - K)^+$ is the payout of a call. Therefore, a replication strategy is to hold the static position $\frac{2}{K^2}dK$ puts of strike $K \le 1$ and$\frac{2}{K^2}dK$ calls of strike $K > 1$, and the dynamic position of $\frac{2}{S_t} - 2$ shares of the stock at time t.

This replication strategy only requires that S is positive and that Itô's formula applies, but is otherwise model-independent. (But don't forget about notions of admissibility, and even more fundamentally, that we need to be in a situation where we can safely ignore bid-ask spread, price impact, transaction costs, etc.)

Problem 5. Consider a market with zero interest rate $r = 0$ and a stock with price dynamics

$$dS_t = S_t \sigma_t dW_t$$

where σ is independent of the $\mathbb{Q}$-Brownian motion W. Let

$$C(T,K) = \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+].$$

(a) Show that there is a family of measures μ_T on $[0,\infty)$ such that

$$C(T,K) = S_0 \int F(v, K/S_0)\mu_T(dv)$$

where F is the function defined in Problem 2.

(b) If there are constants $a \le b$ such that $a \le \sigma_t \le b$ a.s., show that the implied volatility satisfies

$$a \le \Sigma(T,K) \le b.$$

where the implied volatility $\Sigma(T,K)$ is defined implicitly as the unique non-negative solution σ of the equation

$$F(T\sigma^2, K/S_0) = C(T,K)/S_0.$$

(c) Show the equality $\Sigma(T,K) = \Sigma(T, S_0^2/K)$, i.e. the function $x \mapsto \Sigma(T, S_0 e^x)$ is even. Hint: First prove the identity

$$F(v,m) = 1 - m + mF(v, 1/m).$$

Solution 5. (a) Notice that conditional on the process σ, the distribution of

$$\log S_T = \log S_0 - \frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s$$

is normal, since σ and W are independent. Hence

$$\mathbb{E}[(S_T - K)^+|\sigma] = S_0\ F\left(\int_0^T \sigma_s^2 ds, K/S_0\right).$$

The conclusion follows from the tower property of conditional expectations, where the measure μ_T is the law of the non-negative random variable $\int_0^T \sigma_s^2\ ds$.

(b) Since $v \mapsto F(v,m)$ is increasing, we have

$$F(Ta^2, K/S_0) \le \int_{[a,b]} F(v, K/S_0)\mu_T(dv) \le F(Tb^2, K/S_0).$$

But since the middle term is just $F(T\Sigma(T,K)^2, K/S_0)$, we can conclude

$$a \le \Sigma(T,K) \le b.$$

(c) Notice the Black–Scholes call price function satisfies the following identity:

$$\begin{aligned}
F(v,m) &= \Phi(-\log m/\sqrt{v} + \sqrt{v}/2) - m\Phi(-\log m/\sqrt{v} - \sqrt{v}/2) \\
&= 1 - \Phi(\log m/\sqrt{v} - \sqrt{v}/2) - m\left[1 - \Phi(\log m/\sqrt{v} + \sqrt{v}/2)\right] \\
&= 1 - m + m\left[\Phi(\log m/\sqrt{v} + \sqrt{v}/2) - (1/m)\Phi(\log m/\sqrt{v} - \sqrt{v}/2)\right] \\
&= 1 - m + m\ F(v, 1/m)
\end{aligned}$$

Now use the above calculation:

$$\begin{aligned}
S_0\ F(T\Sigma(T,K)^2, K/S_0) &= \int S_0\ F(v, K/S_0)\mu_T(dv) \\
&= \int \left[S_0 - K + K\ F(v, S_0/K)\right]\mu_T(dv) \\
&= S_0 - K + K\ F(T\Sigma(T, S_0^2/K)^2, S_0/K) \\
&= S_0\ F(T\Sigma(T, S_0^2/K)^2, K/S_0).
\end{aligned}$$

In particular, the implied volatility smile in this model (where the volatility is uncorrelated with the driving Brownian motion) is symmetric as a function of log-moneyness $\log(K/S_0)$. This observation is due to Renault and Touzi in 1996.

Problem 6. * (Hull–White extension of Cox–Ingersoll–Ross) Consider the short rate model given by

$$dr_t = \lambda(\bar{r}(t) - r_t)\ dt + \gamma\sqrt{r_t}\ dW_t$$

for positive constants λ and γ and a deterministic function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}$. Find the initial forward rate curve $T \mapsto f_0^T$ for this model.

Solution 6. Consider the PDE

$$\begin{aligned}
&\frac{\partial V}{\partial t}(t,T,r) + \lambda(\bar{r}(t) - r)\frac{\partial V}{\partial r}(t,T,r) + \frac{1}{2}\gamma^2 r\frac{\partial^2 V}{\partial r^2}(t,T,r) = rV(t,T,r) \\
&V(T,T,r) = 1.
\end{aligned}$$

As usual we can make the ansatz

$$V(t,T,r) = e^{-rA(t,T) - B(t,T)}$$

for some functions A and B which satisfy the boundary conditions $A(T,T) = B(T,T) = 0$. Substituting this into the PDE yields

$$-\frac{\partial A}{\partial t}r - \frac{\partial B}{\partial t} - \lambda(\bar{r}(t) - r)A + \frac{\gamma^2}{2}rA^2 = r$$

which yields the coupled system

$$\begin{aligned}
\frac{\partial A}{\partial t}(t,T) &= \lambda A(t) + \frac{\gamma^2}{2}A(t)^2 - 1 \\
\frac{\partial B}{\partial t}(t,T) &= -\lambda\bar{r}(t)A(t,T).
\end{aligned}$$

The equation for A is a Riccati equation, whose solution is

$$\begin{aligned}
A(t,T) &= \frac{2(e^{\beta(T-t)}-1)}{(\beta+\lambda)e^{\beta(T-t)}+(\beta-\lambda)} \\
B(t,T) &= \int_t^T \lambda\bar{r}(s)A(s,T)ds
\end{aligned}$$

where $\beta = \sqrt{\lambda^2+2\gamma^2}$. Hence, the time 0 forward rates are given by

$$f_0^T = -\frac{\partial}{\partial T}\log P_0^T = \frac{4\beta^2 e^{\beta T}}{[(\beta+\lambda)e^{\beta T}+(\beta-\lambda)]^2}r_0 + \int_0^T \frac{4\beta^2 e^{\beta(T-s)}\lambda\bar{r}(s)ds}{[(\beta+\lambda)e^{\beta(T-s)}+(\beta-\lambda)]^2}.$$

Problem 7. Let $W^1,\dots,W^m$ be m independent Brownian motions, and let $X^1,\dots,X^m$ evolve as

$$dX_t^i = aX_t^i\,dt + b\,dW_t^i$$

given initial conditions $X_0^1,\dots,X_0^m$ and fixed constants a,b.

Let

$$Z_t = \sum_{i=1}^m (X_t^i)^2 = \|X_t\|^2.$$

Show that there exists constants α,β,γ and a scalar Brownian motion $\hat{W}$ such that

$$dZ_t = (\alpha Z_t+\beta)dt + \gamma\sqrt{Z_t}d\hat{W}_t.$$

Solution 7. Let $f(x)=\|x\|^2$ so that $\frac{\partial f}{\partial x^i}=2x^i$ and $\frac{\partial^2 f}{\partial x^i\partial x^j}=2$ if $i=j$ and 0 otherwise. By Itô's formula

$$\begin{aligned}
dZ_t &= \sum_{i=1}^n 2X_t^i dX_t^i + \sum_{i=1}^n d\langle X^i\rangle_t \\
&= (aZ_t+nb^2)dt + \sum_{i=1}^n 2bX_t^i dW_t^i
\end{aligned}$$

The conclusion now follows with $\alpha=a$, $\beta=nb^2$ and $\gamma=2b$, by defining

$$\hat{W}_t = \int_0^t \frac{X_s\cdot dW_s}{\|X_s\|}$$

and noting that $\hat{W}$ is a Brownian motion by Lévy's characterisation theorem, since $\langle W\rangle_t = t$.

Problem 8. Given positive constants $\lambda,\bar{r},\gamma$ such that $\bar{r}<\gamma$ and $\lambda\bar{r}>\gamma^2/2$, and an initial condition $0<r_0<\gamma$ and a Brownian motion W, it is possible to show that there exists a process $(r_t)_{t\ge 0}$ satisfying

$$dr_t = \lambda(\bar{r}-r_t)dt + (\gamma-r_t)\sqrt{r_t}dW_t$$

such that $0 < r_t < \gamma$ for all $t \geq 0$ almost surely. Define the function $H : \mathbb{R}_+ \times (0, \gamma) \to (0, 1)$ by

$$\mathbb{E}[e^{-\int_0^t r_s \, ds} | r_0 = r] = H(t, r)$$

Show that $H(t, \cdot)$ is a quadratic function for each $t \geq 0$.

Solution 8. We will show that there is a function $G : \mathbb{R}_+ \times (0, \gamma) \to (0, 1)$ such that $G(t, \cdot)$ is quadratic and furthermore satisfies the PDE

$$\frac{\partial G}{\partial t} = \lambda(\bar{r} - r)\frac{\partial G}{\partial r} + \frac{1}{2}(\gamma - r)^2 r \frac{\partial^2 G}{\partial r^2} - rG$$

with boundary condition $G(0, r) = 1$ for all r. Assuming for the moment that we have such a function G, note that for fixed $t > 0$, the process $(M_s)_{0 \leq s \leq T}$ defined by

$$M_s = e^{-\int_0^s r_u \, du} G(t - s, r_s)$$

is a local martingale by Itô's formula. Furthermore, since M is bounded (because the process $(r_s)_{0 \leq s \leq t}$ is bounded), the local martingale M is actually true martingale. Hence

$$G(t, r_0) = M_0 = \mathbb{E}(M_t) = \mathbb{E}[e^{-\int_0^t r_s \, ds}].$$

That is to say, $G = H$.

Now to find G, we make the ansatz that

$$G(t, r) = g_0(t) + g_1(t)r + g_2(t)r^2$$

with $g_0(0) = 1$ and $g_1(0) = g_2(0) = 0$. Plugging this in yields

$$\begin{aligned} g_0' + g_1' r + g_2' r^2 &= \lambda(\bar{r} - r)(g_1 + 2g_2 r) + (\gamma - r)^2 r g_2 - r(g_0 + g_1 r + g_2 r^2) \\ &= \lambda \bar{r} g_1 - r[g_0 + \lambda g_1 - (2\lambda\bar{r} + \gamma^2)g_2] - r^2[g_1 + 2(\lambda + \gamma)g_2] \end{aligned}$$

(the coefficients are not really important– what one should notice is that the coefficient of r^3 vanishes on the right-hand side). Letting $g = (g_1, g_2, g_3)^\top$, by comparing coefficients, we need only solve the system of linear ODEs

$$g' = Sg, \;\; g(0) = (1, 0, 0)^\top$$

where

$$S = \begin{pmatrix} 0 & \lambda\bar{r} & 0 \\ -1 & -\lambda & 2\lambda\bar{r} + \gamma^2 \\ 0 & -1 & -2(\lambda + \gamma) \end{pmatrix}.$$