Problem 1. Let $Z$ be a homogeneous Markov process taking values in a finite space $E$ with transition probabilities

$$p_{z,y} = \mathbb{P}(Z_1 = y | Z_0 = z) \text{ for } z, y \in E.$$ 

Let $R : E \to (-1, \infty)$ and $F : \mathbb{Z}_+ \times E \to \mathbb{R}^d$ be functions such that

$$\sum_{y \in E} F(t+1, y)p_{z,y} = (1 + R(z))F(t, z) \text{ for all } t \geq 0, z \in E.$$ 

Consider a market with a bank account with risk-free rate $r_t = R(Z_{t-1})$ and $d$ stocks with prices $S_t^i = F^i(t, Z_t)$ for $i = 1, \ldots, d$.

(a) Show that the market has no arbitrage.

Now suppose that for each $z \in E$ that the set

$$\mathcal{Y}(z) = \{y \in E : p_{z,y} > 0\}$$

has exactly $d + 1$ elements. Furthermore, suppose that for all $t > 0$ and $z \in E$, the $d + 1$ functions \{1, $F^1(t, \cdot), \ldots, F^d(t, \cdot)$\} are linearly independent when restricted to $\mathcal{Y}(z)$, where \{1\}(y) = 1 for all $y \in E$.

(b) Show that the market is complete.

(c) Define the $d \times d$ matrix-valued function for $0 < t \leq T, z \in E, y_0 \in \mathcal{Y}(z)$ by

$$\Delta(t, z, y_0) = (F^i(t, y) - F^i(t, y_0) : y \in \mathcal{Y}(z) \setminus \{y_0\}, 1 \leq i \leq d).$$

Show that $\Delta(t, z, y_0)$ is invertible.

(d) For a function $f : E \to \mathbb{R}$ consider the $d \times 1$ vector defined for $z \in E$ and $y_0 \in \mathcal{Y}(z)$ by

$$\tilde{f}(z, y_0) = (f(y) - f(y_0) : y \in \mathcal{Y}(z) \setminus \{y_0\}).$$

Show that the $d \times 1$ vector $\Delta(t, z, y_0)^{-1}\tilde{f}(z, y_0)$ does not depend on $y_0$.

(e) Fix a function $g : E \to \mathbb{R}$ and let $V(T, z) = g(z)$ and

$$V(t, z) = \frac{1}{1 + R(z)} \sum_{y \in E} V(t+1, y)p_{z,y} \text{ for all } 0 \leq t \leq T - 1, z \in E.$$ 

Define a $\mathbb{R}^d$-valued function for $0 < t \leq T, z \in E, y_0 \in \mathcal{Y}(z)$ by

$$\Pi(t, z) = \Delta(t, z, y_0)^{-1}(V(t, y) - V(t, y_0) : y \in \mathcal{Y}(z) \setminus \{y_0\})$$

for any $y_0 \in \mathcal{Y}(z)$. (The choice of $y_0$ is irrelevant by part (d).) Finally, define a real-valued function by

$$\Phi(t, z) = V(t - 1, z) - \Pi(t, z) \cdot F(t - 1, z).$$

Show that the European claim with time $T$ payout $\xi_T = g(S_T)$ is with unique no-arbitrage price $\xi_t = V(t, Z_t)$ and replicating strategy $H = (\phi, \pi)$ where

$$\phi_t = \frac{\Phi(t, Z_{t-1})}{B_{t-1}} \text{ and } \pi_t = \Phi(t, Z_{t-1})$$
Problem 2. Let \((\zeta_{t,T})_{1 \leq t < T}\) be a collection of positive random variables such that \(\zeta_{t,T}\) is \(\mathcal{F}_t\)-measurable for all \(t\) and that
\[
\mathbb{E} \left[ \left( \prod_{u=t+1}^{T} \zeta_{t,u} \right)^{-1} \big| \mathcal{F}_{t-1} \right] = 1
\]
for all \(1 \leq t < T\). Now given a non-random sequence \(f_{0,T} > -1\) for \(T > 0\), let
\[
1 + f_{t,T} = (1 + f_{t-1,T})\zeta_{t,T} \quad \text{for } 1 \leq t < T.
\]
Let \(r_t = f_{t-1,T}\) for \(t \geq 1\) and \(P_{t,T} = \left( \prod_{u=t+1}^{T} (1 + f_{t,u}) \right)^{-1} \) for \(0 \leq t < T\).

Consider a market with a bank account with time \(t\) risk-free interest rate \(r_t\) and a collection of bonds such that \(P_{t,T}\) is the time \(t\) of the bond of maturity \(T\).

(a) Show that the market has no arbitrage.
(b) Use example sheet 2 problem 6 to show that the forward rate at time \(t\) for maturity \(T\) is given by \(f_{t,T}\).
(c) Let \(\zeta_{t,T} = \exp(\sigma_{t,T}^2 + \mu_{t,T})\) where \(\sigma_{t,T}\) and \(\mu_{t,T}\) are \(\mathcal{F}_{t-1}\) measurable and \(\xi_t\) is \(N(0,1)\) and independent of \(\mathcal{F}_{t-1}\). Show that
\[
\mu_{t,T} = \sigma_{t,T} \sum_{u=t+1}^{T-1} \sigma_{u,T}^2 + \frac{1}{2} \sigma_{t,T}^2
\]

Problem 3. Let \(g\) be a function on the integers, and define functions \(g'\) and \(g''\) by the formulae
\[
g'(x) = \frac{1}{2} [g(x+1) - g(x-1)] \quad \text{and} \quad g''(x) = g(x+1) - 2g(x) + g(x-1)
\]
for all integers \(x\).

Let \((x_t)_t\) be a sequence of integers with \(x_t - x_{t-1} \in \{-1,0,1\}\) for each \(t \geq 1\). Show that for all \(t \geq 0\) we have
\[
g(x_t) = g(x_0) + \sum_{s=1}^{t} g'(x_{s-1})(x_s - x_{s-1}) + \frac{1}{2} \sum_{s=1}^{t} g''(x_{s-1})(x_s - x_{s-1})^2.
\]

Problem 4. * Let \((S_t)_{t \geq 0}\) be a discrete-time martingale such that \(S_0\) is an integer and for all \(t \geq 1\) the increment \(S_t - S_{t-1}\) is valued in the set \(\{-1,0,1\}\).

(a) Prove the identity
\[
(S_T - K - 1)^+ - 2(S_T - K)^+ + (S_T - K + 1)^+ = 1_{\{S_T = K\}}
\]
for integers \(K\) and \(T \geq 0\).
(b) Prove the identity
\[
(S_T - K)^+ = (S_0 - K)^+ + \sum_{t=1}^{T} f(S_{t-1} - K)(S_t - S_{t-1}) + \frac{1}{2} \sum_{t=1}^{T} 1_{\{S_t = K\}}(S_t - S_{t-1})^2
\]
for integers \(K\) and \(T \geq 1\), where \(f\) is defined by
\[
f(x) = 1_{\{x > 0\}} + \frac{1}{2} 1_{\{x = 0\}}.
\]

Let
\[
C(T, K) = \mathbb{E}[(S_T - K)^+]
\]
for integers \( K \) and \( T \geq 0 \) and
\[
\sigma^2(T, K) = \text{Var}(S_{T+1}|S_T = K)
\]
for integers \( K \) and \( T \) such that \(|K - S_0| \leq T\).
(c) Using parts (a) and (b), or otherwise, prove the identity
\[
C(T + 1, K) - C(T, K) = \frac{1}{2} \sigma^2(T, K)[C(T, K + 1) - 2C(T, K) + C(T, K - 1)]
\]
for integers \( K \) and \( T \) such that \(|K - S_0| \leq T\).
(d) Comment an application part (c) to finance.

**Problem 5.** Let \( f \) be a positive continuous (non-random) function and \( W \) a Brownian motion. Use Lévy’s characterisation of Brownian motion to show that \( \int_0^t f(s)dW_s \) is a normal random variable with mean zero and variance \( \int_0^t f(s)^2ds \).

**Problem 6.** *(Ornstein–Uhlenbeck process)* Let \( W \) be a Brownian motion, and let
\[
X_t = e^{at}x + b \int_0^t e^{a(t-s)}dW_s
\]
for some \( a, b, x \in \mathbb{R} \).
(a) Verify that \( (X_t)_{t \geq 0} \) satisfies the following stochastic differential equation:
\[
dX_t = aX_t \, dt + b \, dW_t, \quad X_0 = x.
\]
(b) Show that
\[
X_t \sim N \left( e^{at}x, \frac{b^2}{2a}(e^{2at} - 1) \right).
\]
(c) What is the distribution of the random variable \( \int_0^T X_t \, dt \)?

**Problem 7.** Let \( W \) be a Brownian motion. Show that if \( Y_t = W_t^3 - 3tW_t \) then \( Y \) is a martingale (1) by hand, and (2) by Itô’s formula.

**Problem 8.** (Heat equation) Let \( W \) be a scalar Brownian motion, and let \( g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function that satisfy the partial differential equation
\[
\frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} = 0
\]
with terminal condition
\[
g(T, x) = G(x).
\]
(a) Show that \( (g(t, W_t))_{t \in [0, T]} \) is a local martingale.
(b) If the function \( g \) is bounded, deduce the formula
\[
g(t, x) = \int_{-\infty}^{\infty} G(x + \sqrt{T - t}z) \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz.
\]

**Problem 9.** (Strictly local martingale) This is a technical exercise to exhibit a local martingale that is not a true martingale. Let \( W = (W^1, W^2, W^3) \) be a three-dimensional Brownian motion and let \( u = (1, 0, 0) \). It is a fact that \( \mathbb{P}(W_t \neq u \text{ for all } t \geq 0) = 1 \).
(a) Let \( X_t = |W_t - u|^{-1} \). Use Itô’s formula and Lévy’s characterisation of Brownian motion to show that
\[
dX_t = X_t^2dZ_t, \quad X_0 = 1
\]
where \( Z \) is a Brownian motion. In particular, show that \( X \) is a positive local martingale.

(b) By directly evaluating the integral or otherwise, show that
\[
\mathbb{E}(X_t) = 2 \Phi(t^{-1/2}) - 1
\]
for all \( t > 0 \), where \( \Phi \) is the distribution function of a standard normal random variable. Why does this imply that \( X \) is a strictly local martingale?

**Problem 10.** (strictly local martingales again) (a) Suppose that \( X \) is positive martingale with \( X_0 = 1 \). Fix \( T > 0 \) and let
\[
\frac{dQ}{dP} = X_T.
\]
Let \( Y_t = 1/X_t \) for all \( t \geq 0 \). Show that \((Y_t)_{0 \leq t \leq T}\) is a positive martingale under \( Q \).

(b) Continuing from part (a), now suppose that \( X \) has dynamics
\[
dX_t = X_t \sigma_t dW_t
\]
where \( W \) is a Brownian motion under \( P \). Use Girsanov’s theorem to show that there exists a \( Q \)-Brownian motion \( \hat{W} \) such that
\[
dY_t = Y_t \sigma_t d\hat{W}_t
\]
(c) Let \( X \) be a positive local martingale with \( X_0 = 1 \) and dynamics
\[
dX_t = X_t^2 dW_t.
\]
Our goal is to show that \( X \) is a strictly local martingale. For the sake of finding a contradiction, suppose \( X \) is a true martingale. In the notation of parts (a) and (b), show that
\[
P(Y_t > 0) = 1 \text{ but } Q(Y_t > 0) = \Phi(t^{-1/2}).
\]
Why does this contradict the assumption that \( X \) is a true martingale?