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Example sheet 3 - Michaelmas 2010

Problem 1. The auto-covariance function of a square-integrable stochastic process X is defined by

$$c(s,t) = \mathbb{E}(X_s X_t) - \mathbb{E}(X_s)\mathbb{E}(X_t).$$

A stochastic process X is called *Gaussian* if the random variables $X_{t_1}, \ldots, X_{t_n}$ are jointly normal for all $t_1, \ldots, t_n \geq 0$. Prove that X is a Brownian motion if and only if X is a continuous, mean-zero, Gaussian process with auto-covariance function $c(s,t) = s \wedge t$.

Problem 2. * (existence of Brownian motion) Let $Z_1, Z_2, \ldots$ be a sequence of independent $N(0,1)$ random variables. Let H be the Hilbert space of (equivalence classes) of measurable functions $h : [0,1] \to \mathbb{R}$ such that $\int_0^1 h(s)^2 ds < \infty$, and let $(h_n)_{n\geq 1}$ be an orthonormal basis of H. For each n, define continuous functions g_n by

$$g_n(t) = \int_0^t h_n(s)ds,$$

and for each $t \in [0,1]$, define a random variable W_t by

$$W_t = \lim_{N\to\infty} \sum_{n=1}^{N} Z_n g_n(t)$$

where the limit is interpreted in sense of $L^2(\Omega)$. Show that $W_t \sim N(0,t)$ and that $\mathbb{E}(W_s W_t) = s \wedge t$.

[To show the existence of Brownian motion, we must prove that there is an almost sure event $\Omega_0 \subseteq \Omega$ such that $t \mapsto W_t(\omega)$ is continuous for all $\omega \in \Omega_0$. This part of the proof is more difficult.]

Problem 3. Let W be a Brownian motion. Show that the following processes are Browniam motions:

(1) $\left(\frac{1}{c} W_{c^2 t}\right)_{t\in\mathbb{R}_+}$ for a constant $c \neq 0$.
(2) $\left(t W_{1/t}\right)_{t\in\mathbb{R}_+}$
(3) $(W_{t+a} - W_a)_{t\in\mathbb{R}_+}$ for a constant $a > 0$.

Problem 4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous (deterministic) function. Show that the stochastic integral $\int_0^T f(t) dW_t$ is normal with mean zero and variance $\int_0^T f(t)^2 dt$.

Problem 5. (Ornstein–Uhlenbeck process) Let W be a Wiener process, and let

$$X_t = e^{-at} x + b \int_0^t e^{-a(t-s)} dW_s$$

for some $a, b, x \in \mathbb{R}$. Verify that $(X_t)_{t\in\mathbb{R}_+}$ satisfies the following stochastic differential equation:

$$dX_t = -a X_t \, dt + b \, dW_t, \quad X_0 = x.$$

Show that

$$X_t \sim N\left(e^{-at} x, \frac{b^2}{2a}(1 - e^{-2at})\right).$$

Compute $\mathrm{Cov}(X_s, X_t)$. Show that if $a > 0$ then $X_t \to N(0, \frac{b^2}{2a})$ in distribution as $t \to \infty$.

Problem 6. (Stratonovich integral) Let X and Y be Itô processes. The Stratonovich integral of X with respect to Y is defined by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t$$

where the integral on the right side of the equation is an Itô integral. Show that if f is three times continuously differentiable, then Itô's formula can be written

$$df(X_t) = f'(X_t) \circ dX_t.$$

Problem 7. Let M be a positive continuous local martingale. Prove that M is a supermartingale.

Problem 8. Let M be a continuous local martingale such that

$$\mathbb{E}(\sup_{s\in[0,t]} |M_s|) < \infty$$

for all $t \geq 0$. Show that M is actually a true martingale.

Problem 9. Let X be an Itô process and define the stochastic (or Doléans-Dade) exponential $\mathcal{E}(X)$ as the process

$$\mathcal{E}(X)_t = e^{-\frac{1}{2}\langle X\rangle_t + X_t}.$$

Show that $\mathcal{E}(X)$ is an Itô process. Furthermore, show that $\mathcal{E}(X)$ is a local martingale if and only if X is.

Problem 10. Let α be a predictable process such that $|\alpha_t(\omega)| \leq C$ for all (t, ω) and some constant $C > 0$, and let W be a Brownian motion. Without resorting to Novikov's criterion, show that the process $Z_t = \mathcal{E}\left(\int_0^t \alpha_s dW_s\right)$ is a true martingale, where the $\mathcal{E}$ notation is defined in the previous problem.

Problem 11. Let W be a Brownian motion. Show that if $Y_t = W_t^3 - 3tW_t$ then Y is a martingale (1) by hand, and (2) by Itô's formula.

Problem 12. (Heat equation) Let W be a scalar Brownian motion, and let $h : [0, T]\times\mathbb{R} \to \mathbb{R}$ satisfy the partial differential equation

$$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = 0$$

with terminal condition

$$h(T, x) = G(x).$$

Show that $H = (h(t, W_t))_{t\in[0,T]}$ is a local martingale. Suppose that there are constants $A, B > 0$ such that

$$|G(x)| \leq Ae^{B|x|}$$

and

$$|\frac{\partial}{\partial x}h(t, x)| \leq Ae^{B|x|}$$

for all $(t, x) \in [0, T] \times \mathbb{R}$. Show that H is a true martingale. Deduce the formula

$$h(t, x) = \int_{-\infty}^{\infty} G(x + \sqrt{T - t}z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz.$$

Problem 13. (square-root diffusion) Let W be an n-dimensional Brownian motion, and define an n-dimensional process X to be the solution to the SDE

$$dX_t = -X_t dt + dW_t$$

with $X_0 = x \in \mathbb{R}^n$. If $R_t = |X_t|^2$, show that there exists a scalar Brownian motion Z such that

$$dR_t = (n - 2R_t)dt + 2\sqrt{R_t} dZ_t.$$

Problem 14. (Strict local martingale) This is a technical exercise to exhibit a local martingale that is not a true martingale. Let L be a local martingale with $L_0 = 1$ satisfying the following stochastic differential equation

$$dL_t = L_t^2 dW_t.$$

Suppose for the sake of finding a contradiction, that L is a positive martingale. If L is defined on $(\Omega, \mathcal{F}, \mathbb{P})$, define the locally equivalent measure $\mathbb{Q}$ via the density process L. Let $B_t = 1/L_t - 1$. Show that B is a Brownian motion under $\mathbb{Q}$. This is the contradiction, as $\mathbb{P}(L_t < 0) = 0$ by supposition, but $\mathbb{Q}(B_t < -1) = \mathbb{Q}(L_t < 0) > 0$.