Advanced Financial Models Michael Tehranchi
Example sheet 3 - Michaelmas 2010

Problem 1. The auto-covariance function of a square-integrable stochastic process X is defined by

$$c(s,t) = \mathbb{E}(X_s X_t) - \mathbb{E}(X_s)\mathbb{E}(X_t).$$

A stochastic process X is called *Gaussian* if the random variables $X_{t_1}, \ldots, X_{t_n}$ are jointly normal for all $t_1, \ldots, t_n \geq 0$. Prove that X is a Brownian motion if and only if X is a continuous, mean-zero, Gaussian process with auto-covariance function $c(s,t) = s \wedge t$.

Solution 1. Suppose X is a Brownian motion. Then, by definition, it is a continuous, mean-zero, Gaussian process. Its auto-covariance function is given by the computation:

$$\mathbb{E}(X_s X_t) = \mathbb{E}[X_s(X_t - X_s) + X_s^2] = s$$

if $0 \leq s \leq t$, since the increments X_s and $X_t - X_s$ are independent and mean-zero.

Conversely, suppose X is a continuous, mean-zero, Gaussian process with auto-covariance function $c(s,t) = s \wedge t$. Then the covariance of the increments $X_t - X_s$ and $X_v - X_u$ where $0 \leq s \leq t \leq u \leq v$ is given by

$$\mathbb{E}[(X_t - X_s)(X_v - X_u)] = t \wedge v - s \wedge v - t \wedge u + s \wedge u = 0$$

Since jointly normal random variables are uncorrelated if and only if they are independent, we can conclude from the above computation that the increments of X are independent, proving that X is a Brownian motion.

Problem 2. * (existence of Brownian motion) Let $Z_1, Z_2, \ldots$ be a sequence of independent $N(0,1)$ random variables. Let H be the Hilbert space of (equivalence classes) of measurable functions $h : [0,1] \to \mathbb{R}$ such that $\int_0^1 h(s)^2 ds < \infty$, and let $(h_n)_{n \geq 1}$ be an orthonormal basis of H. For each n, define continuous functions g_n by

$$g_n(t) = \int_0^t h_n(s) ds,$$

and for each $t \in [0,1]$, define a random variable W_t by

$$W_t = \lim_{N \to \infty} \sum_{n=1}^{N} Z_n g_n(t)$$

where the limit is interpreted in sense of $L^2(\Omega)$. Show that $W_t \sim N(0,t)$ and that $\mathbb{E}(W_s W_t) = s \wedge t$.

[To show the existence of Brownian motion, we must prove that there is an almost sure event $\Omega_0 \subseteq \Omega$ such that $t \mapsto W_t(\omega)$ is continuous for all $\omega \in \Omega_0$. This part of the proof is more difficult.]

Solution 2. First we need a little lemma: If $X_n \sim N(\mu_n, \sigma_n^2)$ and $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$, then $X_n \to X$ in distribution, where $X \sim N(\mu, \sigma^2)$. The proof is simple: The distribution function $F_n(x) = \Phi\left(\frac{x-\mu_n}{\sigma_n}\right)$ converges pointwise to $\Phi\left(\frac{x-\mu}{\sigma}\right)$.

Now, for each N, let

$$W_t^{(N)} = \sum_{n=1}^{N} Z_n g_n(t)$$

Then W_t^N is normally distributed with mean 0 and variance

$$\mathbb{E}[(W_t^{(N)})^2] = \sum_{n=1}^{N} g_n(t)^2.$$

Hence W_t is normal with mean 0 and variance

$$\begin{aligned}
\mathbb{E}[W_t^2] &= \sum_{n=1}^{\infty} g_n(t)^2 \\
&= \sum_{n=1}^{\infty} \langle \mathbb{1}_{[0,t]}, h_n \rangle^2 \\
&= \| \mathbb{1}_{[0,t]} \|^2 \\
&= t
\end{aligned}$$

by Parseval's identity, where the inner product on H is denoted $\langle f, g \rangle = \int_0^1 f(s)g(s)ds$ with corresponding norm $\|f\| = \sqrt{\langle f, f \rangle}$.

Similarly,

$$\begin{aligned}
\mathbb{E}[W_s W_t] &= \sum_{n=1}^{\infty} g_n(s) g_n(t) \\
&= \sum_{n=1}^{\infty} \langle \mathbb{1}_{[0,s]}, h_n \rangle \langle \mathbb{1}_{[0,t]}, h_n \rangle \\
&= \langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \rangle \\
&= s \wedge t.
\end{aligned}$$

[Since $t \mapsto W_t^{(N)}(\omega)$ is continuous, the final step in the proof of the existence of Brownian motion is to show that for all ω in some almost sure set Ω_0, the convergence $W_t^{(N)}(\omega) \to W_t(\omega)$ is uniform.]

Problem 3. Let W be a Brownian motion. Show that the following processes are Browniam motions:

(1) $\left(\frac{1}{c} W_{c^2 t}\right)_{t \in \mathbb{R}_+}$ for a constant $c \neq 0$.
(2) $\left(t W_{1/t}\right)_{t \in \mathbb{R}_+}$
(3) $(W_{t+a} - W_a)_{t \in \mathbb{R}_+}$ for a constant $a > 0$.

Solution 3. In each case, the given process is a continuous,[1] mean-zero Gaussian process. We need only show the autocovariance $\mathrm{Cov}(Z_s, Z_t) = s \wedge t$ is correct.

[1]except in case (2) where continuity at $t = 0$ is not obvious. We must prove

$$\lim_{t \downarrow 0} t W_{1/t} = \lim_{t \uparrow \infty} \frac{W_t}{t} = 0 \text{ almost surely.}$$

(1) Let $Z_t = \frac{1}{c} W_{c^2 t}$. Then $\mathbb{E}(Z_t) = 0$ and
$$\mathbb{E}(Z_s Z_t) = \frac{1}{c^2} \min\{c^2 s, c^2 t\} = s \wedge t.$$
(2) Let $Z_t = t W_{1/t}$ for $t > 0$ and $Z_0 = 0$. Then $\mathbb{E}(Z_t) = 0$ and
$$\mathbb{E}(Z_s Z_t) = st \min\{1/s, 1/t\} = s \wedge t.$$
(3) Let $Z_t = W_{t+a} - W_a$. Then $\mathbb{E}(Z_t) = 0$ and
$$\mathbb{E}(Z_s Z_t) = (t+a) \wedge (s+a) - a \wedge (t+a) - a \wedge (s+a) + a \wedge a = s \wedge t.$$

Problem 4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous (deterministic) function. Show that the stochastic integral $\int_0^T f(t) dW_t$ is normal with mean zero and variance $\int_0^T f(t)^2 dt$.

Solution 4. Let f_N be a piece-wise constant left-continuous function such that $f_N(s) \to f(s)$ for all $s \in [0, t]$. For instance, we can take $f_N(s) = f(t_{n-1}^{(N)})$ whenever $s \in (t_{n-1}^{(N)}, t_n^{(N)}]$, where $t_n^{(N)} = tn/N$. Then
$$\int_0^t f_N(s) dW_s = \sum_{n=1}^N f_N(t_n^{(N)})(W_{t_n^{(N)}} - W_{t_{n-1}^{(N)}})$$
is Gaussian as it is the sum of independent Gaussians, and converges in L^2 to $\int_0^t f(s) dW_s$. Now, the variance converges
$$\mathbb{E}\left[\left(\int_0^t f_N(s) dW_s\right)^2\right] = \sum_{n=1}^N f_N(t_n^{(N)})^2 (t_n^{(N)} - t_{n-1}^{(N)}) \to \int_0^t f(s)^2 ds$$
by standard theorems of Riemann integration.

Problem 5. (Ornstein–Uhlenbeck process) Let W be a Wiener process, and let
$$X_t = e^{-at} x + b \int_0^t e^{-a(t-s)} dW_s$$

Here's a proof: Fix $\epsilon > 0$.
$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in [2^n, 2^{n+1}]} \frac{|W_t|}{t} > \epsilon\right) &\leq \frac{\mathbb{E}\left(\sup_{t \in [2^n, 2^{n+1}]} \frac{W_t^2}{t^2}\right)}{\epsilon^2} \quad \text{Chebychev's inequality} \\
&\leq \frac{\mathbb{E}\left(\sup_{t \in [2^n, 2^{n+1}]} W_t^2\right)}{2^{2n} \epsilon^2} \\
&\leq \frac{\mathbb{E}\left(\sup_{t \in [0, 2^{n+1}]} W_t^2\right)}{2^{2n} \epsilon^2} \\
&\leq \frac{4\mathbb{E}\left(W_{2^{n+1}}^2\right)}{2^{2n} \epsilon^2} \quad \text{Doob's maximal inequality} \\
&= 2^{-(n-3)} \epsilon^{-2}
\end{aligned}$$
Since the right-hand side is summable, the first Borel–Cantelli lemma implies
$$\mathbb{P}\left(\sup_{t \in [2^n, 2^{n+1}]} \frac{|W_t|}{t} < \epsilon \text{ eventually}\right) = 1 \Rightarrow \mathbb{P}\left(\limsup_{t \uparrow \infty} \frac{|W_t|}{t} < \epsilon\right) = 1.$$
Now take $\epsilon \downarrow 0$.

for some $a, b, x \in \mathbb{R}$. Verify that $(X_t)_{t\in\mathbb{R}_+}$ satisfies the following stochastic differential equation:

$$dX_t = -aX_t\ dt + b\ dW_t, \quad X_0 = x.$$

Show that

$$X_t \sim N\left(e^{-at}x, \frac{b^2}{2a}(1-e^{-2at})\right).$$

Compute $\mathrm{Cov}(X_s, X_t)$. Show that if $a > 0$ then $X_t \to N(0, \frac{b^2}{2a})$ in distribution as $t \to \infty$.

Solution 5. Since $X_t = e^{-at}x + be^{-at}\int_0^t e^{as}dW_s$ we can apply Itô's formula

$$\begin{aligned} dX_t &= -ae^{-at}xdt - abe^{-at}\left(\int_0^t e^{as}dW_s\right)dt + be^{-at}e^{at}dW_t \\ &= -a\left(e^{-at}x + be^{-at}\int_0^t e^{ast}dW_s\right)dt + b\ dW_t \\ &= -aX_tdt + b\ dW_t \end{aligned}$$

Since

$$\int_0^t (e^{-a(t-s)})^2 ds = \frac{1-e^{-2at}}{2a}$$

the first part follows from the previous problem.

Now suppose $s \le t$. Then

$$\begin{aligned} \mathbb{E}(X_sX_t) &= \mathbb{E}\left[\left(\int_0^t b\mathbb{1}_{(0,s]}(u)e^{-a(s-u)}dW_u\right)\left(\int_0^t be^{-a(t-u)}dW_u\right)\right] \\ &= \mathbb{E}\left(\int_0^t b^2\mathbb{1}_{(0,s]}(u)e^{-a(s-u)}e^{-a(t-u)}du\right) \\ &= \frac{b^2}{2a}(e^{-a(t-s)} - e^{-a(s+t)}) \end{aligned}$$

so $\mathrm{Cov}(X_s, X_t) = \frac{b^2}{2a}(e^{-a|t-s|} - e^{-a(s+t)})$ for all s, t.

The last part follows from the observation that X_t is normally distributed and $\mathbb{E}(X_t) \to 0$ and $\mathrm{Var}(X_t) \to \frac{b^2}{2a}$.

Problem 6. (Stratonovich integral) Let X and Y be Itô processes. The Stratonovich integral of X with respect to Y is defined by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t$$

where the integral on the right side of the equation is an Itô integral. Show that if f is three times continuously differentiable, then Itô's formula can be written

$$df(X_t) = f'(X_t) \circ dX_t.$$

Solution 6. By the definition of the Stratonovich integral

$$f'(X_t) \circ dX_t \quad = \quad f'(X_t)dX_t + \frac{1}{2}d\langle f'(X), X\rangle_t.$$

Now applying Itô's formula to $f'(X_t)$ yields

$$df'(X_t) \quad = \quad f''(X_t)dX_t + \frac{1}{2}f'''(X_t)d\langle X\rangle_t$$

and, in particular,

$$d\langle f'(X), X\rangle_t = f''(X_t)d\langle X\rangle_t.$$

We have shown

$$f'(X_t) \circ dX_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t,$$

but the right-hand side is just Itô's formula applied to $f(X_t)$.

Problem 7. Let M be a positive continuous local martingale. Prove that M is a supermartingale.

Solution 7. By definition, there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \to \infty$ almost surely and the stopped process $(M_{t\wedge\tau_n})_{t\in\mathbb{R}_+}$ is a positive martingale. Hence, by (the conditional version of) Fatou's lemma we have the inequality

$$\begin{aligned} \mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}(\lim_{n\to\infty} M_{t\wedge\tau_n}|\mathcal{F}_s) \\ &\leq \liminf_{n\to\infty} \mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) = M_s. \end{aligned}$$

Problem 8. Let M be a continuous local martingale such that

$$\mathbb{E}(\sup_{s\in[0,t]} |M_s|) < \infty$$

for all $t \geq 0$. Show that M is actually a true martingale.

Solution 8. If $(M_t)_{t\in\mathbb{R}_+}$ is a continuous local martingale, then there exists an increasing sequence of stopping times with $\tau_n \uparrow \infty$ almost surely such that for each $n \in \mathbb{N}$, the stopped process $(M_{t\wedge\tau_n})_{t\in\mathbb{R}_+}$ is a martingale.

Since $|M_{t\wedge\tau_n}| \leq \sup_{s\in[0,t]} |M_s|$ for all $n \in \mathbb{N}$ and since $\sup_{s\in[0,t]} |M_s|$ is integrable by assumption, we have the following calculation

$$\begin{aligned} \mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}(\lim_{n\to\infty} M_{t\wedge\tau_n}|\mathcal{F}_s) \\ &= \lim_{n\to\infty} \mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) \\ &= \lim_{n\to\infty} M_{s\wedge\tau_n} \\ &= M_s \end{aligned}$$

for $0 \leq s \leq t$ by the conditional dominated convergence theorem.

The question does not ask, but why is $\sup_{s\in[0,t]} |M_s|$ a random variable, i.e. measurable? The answer is that since M is assumed continuous, the supremum can be realized as the supremum over the countable set $[0, t] \cap Q$, where Q is any countable dense set, for instance, the set of rationals.

Problem 9. Let X be an Itô process and define the stochastic (or Doléans-Dade) exponential $\mathcal{E}(X)$ as the process

$$\mathcal{E}(X)_t = e^{-\frac{1}{2}\langle X\rangle_t + X_t}.$$

Show that $\mathcal{E}(X)$ is an Itô process. Furthermore, show that $\mathcal{E}(X)$ is a local martingale if and only if X is.

Solution 9. Itô's formula yields

$$d\mathcal{E}\left(X\right)_t = \mathcal{E}\left(X\right)_t dX_t.$$

Hence, if X is a local martingale, then $\mathcal{E}(X)$ is too. On the other hand, we have

$$dX_t = \mathcal{E}\left(X\right)_t^{-1} d\mathcal{E}\left(X\right)_t$$

so that if $\mathcal{E}(X)$ is a local martingale, so is X.

Problem 10. Let α be a predictable process such that $|\alpha_t(\omega)| \leq C$ for all (t, ω) and some constant $C > 0$, and let W be a Brownian motion. Without resorting to Novikov's criterion, show that the process $Z_t = \mathcal{E}\left(\int_0^t \alpha_s dW_s\right)$ is a true martingale, where the $\mathcal{E}$ notation is defined in the previous problem.

Solution 10. First we note that if X is a local martingale, the $\mathcal{E}(X)$ is a positive local martingale. (problem 9) Hence, X is a supermartingale. (problem 7) In particular $\mathbb{E}[\mathcal{E}(X)_t] \leq 1$ for all $t \geq 0$.

Now by Itô's formula, we have

$$Z_t = 1 + \int_0^t Z_s \alpha_s dW_s.$$

To show that Z is a true martingale, it is sufficient to prove the inequality

$$\mathbb{E} \int_0^t Z_s^2 \alpha_s^2 ds < \infty$$

for all $t \geq 0$. But

$$\begin{aligned}
\mathbb{E}(\alpha_s^2 Z_s^2) &= \mathbb{E}\left(\alpha_s^2 e^{2\int_0^s \alpha_u dW_u - \int_0^s \alpha_u^2 du}\right) \\
&= \mathbb{E}\left[\alpha_s^2 e^{\int_0^s \alpha_u^2 du} \mathcal{E}\left(\int_0^s 2\alpha_u dW_u\right)\right] \\
&\leq C^2 e^{sC^2} \mathbb{E}\left[\mathcal{E}\left(\int_0^s 2\alpha_u dW_u\right)\right] \\
&\leq C^2 e^{sC^2}
\end{aligned}$$

since $|\alpha_s(\omega)| \leq C$ for all (s, ω).

Problem 11. Let W be a Brownian motion. Show that if $Y_t = W_t^3 - 3tW_t$ then Y is a martingale (1) by hand, and (2) by Itô's formula.

Solution 11. By hand: Since Gaussian random variables have finite moments of all orders, Y is integrable. Indeed, we have

$$\mathbb{E}(|Y_t|) \leq \mathbb{E}(|W_t^3|) + 3t\mathbb{E}(|W_t|) = 5\sqrt{\frac{2t^3}{\pi}} < \infty$$

and

$$\begin{aligned}
\mathbb{E}(Y_t|\mathcal{F}_s) &= \mathbb{E}(W_t^3 - 3tW_t|\mathcal{F}_s) \\
&= \mathbb{E}[(W_t - W_s + W_s)^3 - 3t(W_t - W_s + W_s)|\mathcal{F}_s] \\
&= \mathbb{E}[(W_t - W_s)^3] + 3\mathbb{E}[(W_t - W_s)^2]W_s + 3\mathbb{E}(W_t - W_s)W_s^2 + W_s^3 \\
&\quad -3t\mathbb{E}(W_t - W_s) - tW_s \\
&= 0 + 3(t-s)W_s + 0 + W_s^3 + 0 - tW_s \\
&= Y_s
\end{aligned}$$

for $0 \le s < t$.

By Itô's rule:

$$\begin{aligned}
dY_t &= d(W_t^3 - 3tW_t) \\
&= 3W_t^2\, dW_t + 3W_t\, dt - 3t\, dW_t - 3W_t\, dt \\
&= 3(W_t^2 - t)dW_t
\end{aligned}$$

Recall that if $\mathbb{E}\left(\int_0^t \alpha_s^2\, ds\right) < \infty$ for all $t \ge 0$ then the process $\left(\int_0^t \alpha_s\, dW_s\right)_{t\in\mathbb{R}_+}$ is a martingale. Again, it's clear that the integrand is square integrable in this case, again since Guassian random variables have finite moments of all orders. But, just to be explicit,

$$\mathbb{E}\int_0^t [3(W_s^2 - s)]^2 ds = 9\int_0^t \mathbb{E}(W_s^4 - 2W_s s + s^2)ds = 9\int_0^t 2s^2\, ds = 6t^3 < \infty$$

and hence $(Y_t)_{t\in\mathbb{R}_+}$ is a martingale.

Problem 12. (Heat equation) Let W be a scalar Brownian motion, and let $h : [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfy the partial differential equation

$$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = 0$$

with terminal condition

$$h(T,x) = G(x).$$

Show that $H = (h(t,W_t))_{t\in[0,T]}$ is a local martingale. Suppose that there are constants $A, B > 0$ such that

$$|G(x)| \le Ae^{B|x|}$$

and

$$|\frac{\partial}{\partial x}h(t,x)| \le Ae^{B|x|}$$

for all $(t,x) \in [0,T]\times\mathbb{R}$. Show that H is a true martingale. Deduce the formula

$$h(t,x) = \int_{-\infty}^{\infty} G(x + \sqrt{T-t}\,z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz.$$

Solution 12. By Itô's formula:

$$\begin{aligned} h(t,W_t) &= h(0,0) + \int_0^t \frac{\partial h}{\partial t}(s,W_s)ds + \int_0^t \frac{\partial h}{\partial x}(s,W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 h}{\partial^2 x}(s,W_s)d\langle W\rangle_s = \\ &= h(0,0) + \int_0^t \left[\frac{\partial h}{\partial t}(s,W_s) + \frac{1}{2}\frac{\partial^2 h}{\partial^2 x}(s,W_s)\right]ds + \int_0^t \frac{\partial h}{\partial x}(s,W_s)dW_s \\ &= h(0,0) + \int_0^t \frac{\partial h}{\partial x}(s,W_s)dW_s \end{aligned}$$

Therefore, H is equal to a constant plus a stochastic integral against Brownian motion, and is thus a local martingale.

To show H is a true martingale, it is enough to show

$$\mathbb{E}\left[\int_0^t \left(\frac{\partial h}{\partial x}(s,W_s)\right)^2 ds\right] < \infty.$$

Now if we assume the exponential bound on $\frac{\partial h}{\partial x}$ then we have

$$\begin{aligned} \mathbb{E}\left[\left(\frac{\partial h}{\partial x}(s,W_s)\right)^2\right] &\leq A^2\mathbb{E}\left[e^{2B|W_s|}\right] \\ &= A^2\mathbb{E}\left[e^{2B\sqrt{s}|Z|}\right] \text{ where } Z \sim N(0,1) \\ &< A^2\mathbb{E}\left[e^{2B\sqrt{s}Z} + e^{-2B\sqrt{s}Z}\right] \\ &= 2A^2e^{2B^2 s} \end{aligned}$$

proving that H is a true martingale. In particular,

$$\begin{aligned} h(t,W_t) &= \mathbb{E}[H(T,W_T)|\mathcal{F}_t] \\ &= \mathbb{E}[G(W_T)|\mathcal{F}_t] \\ &= \mathbb{E}[G(W_t + W_T - W_t)|\mathcal{F}_t] \\ &= \int_{-\infty}^{\infty} G(W_t + \sqrt{T-t}z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz. \end{aligned}$$

since the increment $W_T - W_t$ is $N(0,T-t)$ and independent of $\mathcal{F}_t$.

Problem 13. (square-root diffusion) Let W be an n-dimensional Brownian motion, and define an n-dimensional process X to be the solution to the SDE

$$dX_t = -X_t dt + dW_t$$

with $X_0 = x \in \mathbb{R}^n$. If $R_t = |X_t|^2$, show that there exists a scalar Brownian motion Z such that

$$dR_t = (n - 2R_t)dt + 2\sqrt{R_t}dZ_t.$$

Solution 13. Let $f(x) = |x|^2$ so that $\frac{\partial f}{\partial x^i} = 2x^i$ and $\frac{\partial^2 f}{\partial x^i \partial x^j} = 2$ if $i = j$ and 0 otherwise. By Itô's formula

$$\begin{aligned} dR_t &= \sum_{i=1}^n 2X_t^i dX_t^i + \sum_{i=1}^n d\langle X^i \rangle_t \\ &= (n - 2R_t)dt + \sum_{i=1}^n 2X_t^i dW_t^i \end{aligned}$$

The conclusion now follows by defining

$$Z_t = \int_0^t \frac{X_s \cdot dW_s}{|X_s|}$$

and noting that Z is a Brownian motion by Lévy's characterization theorem, since $\langle Z \rangle_t = t$.

Problem 14. (Strict local martingale) This is a technical exercise to exhibit a local martingale that is not a true martingale. Let L be a local martingale with $L_0 = 1$ satisfying the following stochastic differential equation

$$dL_t = L_t^2 dW_t.$$

Suppose for the sake of finding a contradiction, that L is a positive martingale. If L is defined on $(\Omega, \mathcal{F}, \mathbb{P})$, define the locally equivalent measure $\mathbb{Q}$ via the density process L. Let $B_t = 1/L_t - 1$. Show that B is a Brownian motion under $\mathbb{Q}$. This is the contradiction, as $\mathbb{P}(L_t < 0) = 0$ by supposition, but $\mathbb{Q}(B_t < -1) = \mathbb{Q}(L_t < 0) > 0$.

Solution 14. By Itô's formula, we have

$$dB_t = -\frac{1}{L_t^2} dL_t + \frac{1}{L_t^3} d\langle L \rangle_t = -dW_t + L_t dt.$$

If L were a true martingale, then $\hat{W}_t = W_t - \int_0^t L_s \, ds$ would define a Brownian motion under locally eqivalent measure $\mathbb{Q}$ by Girsanov's theorem. And if $\hat{W}$ is Brownian motion, so is $B = -\hat{W}$. But $\mathbb{Q}(B_t < -1) = \Phi(-1/\sqrt{t}) > 0$ contradicting $\mathbb{P}(L_t > 0) = 1$.