Advanced Financial Models Michael Tehranchi
Example sheet 2 - Michaelmas 2019

Problem 1. Consider a one-period market model with no dividends. For the sake of this problem, call an adapted real-valued process $Z = (Z_t)_{t\in\{0,1\}}$ an 'anti-martingale deflator' iff

- $Z_0 \geq 0$, $Z_1 \geq 0$ almost surely and $\mathbb{P}(Z_0 = 0 = Z_1) < 1$,
- $Z_1 P_1$ is integrable and $\mathbb{E}(Z_1 P_1) = -Z_0 P_0$

Show that if there exists a numéraire portfolio, then there does not exist an anti-martingale deflator.

Problem 2. What are the economically appropriate definitions of numéraire portfolio and equivalent martingale measure in the case where the assets may pay a dividend?

Problem 3. Consider a one-period market with three assets. The first asset is a riskless asset with risk-free rate r. The second asset is a stock with prices $(S_t)_{t\in\{0,1\}}$. The third is a contingent claim on the stock with time 1 price $\xi_1 = g(S_1)$, where the function g is convex. Show that if there is no arbitrage, then $\xi_0 \geq \frac{1}{1+r} g[(1+r)S_0]$. Assuming $\xi_0 < \frac{1}{1+r} g[(1+r)S_0]$, find an arbitrage explicitly.

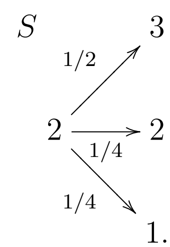
Hint: By the convexity of g, there exists a function λ such that $g(x) \geq g(y) + \lambda(x)(x-y)$ for all $x, y \in \mathbb{R}$.

Problem 4. (Bayes's formula) Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$ with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-field. Prove the identity:

$$\mathbb{E}^{\mathbb{Q}}(X|\mathcal{G}) = \frac{\mathbb{E}^{\mathbb{P}}(ZX|\mathcal{G})}{\mathbb{E}^{\mathbb{P}}(Z|\mathcal{G})}$$

for each random variable X such that X is $\mathbb{Q}$-integrable.

Problem 5. * Consider a trinomial two-asset model with prices $P = (B, S)$ where $B_0 = B_1 = 1$ and S is given by



Find all risk-neutral measures for this model. Now introduce a call option with payout $\xi_1 = (S_1 - 2)^+$. Show that there is an open interval I such that the augmented market (B, S, ξ) has no arbitrage if and only if $\xi_0 \in I$.

Problem 6. Consider an arbitrage-free bond market. Let P_t^T be the price of the bond of maturity T at time t, where $1 \leq t \leq T$. Let the spot rate be $r_t = \frac{1}{P_{t-1}^t} - 1$ and the bank account be $B_t = \prod_{s=1}^t (1 + r_s)$ for all $t \geq 1$ as usual.
(a) Let $\mathbb{Q}$ be a risk-neutral measure, i.e. an equivalent martingale measure relative to the bank account. Show that

$$P_t^T = B_t \, \mathbb{E}^{\mathbb{Q}}(B_T^{-1}|\mathcal{F}_t)$$

for all $0 \leq t \leq T$.

(b) Consider a European contingent claim with maturity T and payout r_T. Show that this claim can be replicated by trading in bonds.
(c) Consider a forward contract initiated at time t for for the payout at time T of r_T. The forward interest rate f_t^T at time t for maturity T is defined to be the forward price of this payout. Show that

$$f_t^T = \frac{P_t^{T-1}}{P_t^T} - 1$$

(d) Show that if the spot rate is not random, then $f_t^T = r_T$.
(e) Let $\mathbb{Q}^T$ be a T-forward measure, i.e. an equivalent martingale measure relative to the bond of maturity T. Show that the forward rate process $(f_t^T)_{0 \leq t < T}$ is a $\mathbb{Q}^T$ martingale.
(f) The quantity

$$y_t^T = (P_t^T)^{-\frac{1}{T-t}} - 1$$

is called the yield at time t of the bond maturing at time T.

Show that the following are equivalent

(1) $f_t^T \geq y_t^T$ a.s. for all $0 \leq t < T$
(2) $T \mapsto y_t^T$ is non-decreasing a.s. for all $t \geq 0$.

(g) Show that the following are equivalent:

(1) $r_t \geq 0$ a.s. for all $t \geq 1$
(2) $t \mapsto B_t$ is non-decreasing a.s.
(3) $T \mapsto P_t^T$ is non-increasing a.s. for each $t \geq 0$
(4) $f_t^T \geq 0$ a.s for all $0 \leq t < T$.
(5) $y_t^T \geq 0$ a.s. for all $0 \leq t < T$.
(6) each martingale deflator is a supermartingale.

Problem 7. (a) Let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{E}(X_n) = 1$ for all n. Use the Borel–Cantelli lemma to show

$$\limsup_{n \to \infty} X_n^{1/n} \leq 1 \text{ a.s.}$$

(b) Consider a bond market as in problem 6. The *long rate* at time t is defined as $\ell_t = \lim_{T \to \infty} y_t^T$ whenever the limit exists.

Suppose that bonds are priced according to the formula in 6(a) for a fixed risk-neutral measure, and that the long rate exists a.s. at all times. Show that the long rate is non-decreasing, that is

$$\ell_s \leq \ell_t \text{ a.s. for all } 0 \leq s \leq t,$$

a fact first discovered by Dybvig, Ingersoll & Ross in 1996.

Problem 8. Let S be a positive supermartingale. Show that there is a positive non-decreasing predictable process A and a positive martingale M such that $A_0 = M_0 = 1$ and $S_t = S_0 M_t / A_t$ for all $t \geq 0$.

Problem 9. * Let $(Y_t)_{0 \leq t \leq T}$ be a given adapted, integrable process, and let $(U_t)_{0 \leq t \leq T}$ be its Snell envelope.
(a) Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is submartingale, then $U_t = \mathbb{E}(Y_T | \mathcal{F}_t)$.
(b) Let τ be any stopping time taking values in $\{0, \ldots, T\}$. Show that the process $(U_{t \wedge \tau})_{0 \leq t \leq T}$ is a supermartingale.

(c) Define the random time τ_* by

$$\tau_* = \min \{t \in \{0, \ldots, T\} : U_t = Y_t\} .$$

Show that τ_* is a stopping time. Furthermore, show that the process $(U_{t\wedge\tau_*})_{t\in\{0,\ldots,T\}}$ is a martingale and, in particular, $U_0 = \mathbb{E}(Y_{\tau_*})$. (That is, τ_* is an optimal stopping time, possibly different than τ^* defined in lectures.)

Problem 10. Let $(X_k)_{k\in K}$ be a collection of real-valued random variables, where K is an arbitrary (possibly uncountable) index set. Our aim is to show there exists a random variable Y taking values in $\mathbb{R} \cup \{+\infty\}$ such that

- $Y \geq X_k$ almost surely for all $k \in K$, and
- if $Z \geq X_k$ almost surely for all $k \in K$ then $Z \geq Y$ almost surely.

This will show that the $Y = \text{ess sup}_k X_k$ exists.
(a) Show that there is no loss assuming that $|X_k(\omega)| \leq 1$ for all (k, ω). Hint: Consider $\tilde{X}_k = \tan^{-1}(X_k)$.

From now on, assume $|X_k(\omega)| \leq 1$ for all (k, ω). Let $\mathcal{C}$ be the collection of all countable subsets of K. Let

$$x = \sup_{A\in\mathcal{C}} \mathbb{E}[\sup_{k\in A} X_k]$$

Let $A_n \in \mathcal{C}$ be such that $\mathbb{E}[\sup_{k\in A_n} X_k] > x - 1/n$ and let $B = \cup_n A_n$. Let $Y = \sup_{k\in B} X_k$.
(b) Why is Y a random variable, i.e. measurable? Show that $\mathbb{E}(Y) = x$.
(c) Pick a $k \in K$, and let $Y_k = \max\{Y, X_k\} = \sup_{h\in B\cup\{k\}} X_h$. Show that $\mathbb{E}(Y_k) = x$. Why does this imply that $X_k \leq Y$ almost surely?
(d) Let Z be a random variable such that $Z \geq X_k$ a.s. for all $k \in K$. Prove that $Z \geq Y$ a.s.