Advanced Financial Models Michael Tehranchi

Example sheet 2 - Michaelmas 2010

Problem 1. (Bayes's formula) Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$ with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-field. Prove the identity:

$$\mathbb{E}^{\mathbb{Q}}(X|\mathcal{G}) = \frac{\mathbb{E}^{\mathbb{P}}(ZX|\mathcal{G})}{\mathbb{E}^{\mathbb{P}}(Z|\mathcal{G})}$$

for each bounded random variable X.

Problem 2. Prove the tower property of conditional expectations.

Problem 3. Suppose $X = (X_t)_{t\in\mathbb{Z}_+}$ has the following properties: For all $t \in \mathbb{Z}_+$

- $\mathbb{E}(|X_t|) < \infty$, and
- $\mathbb{E}(X_{t+1}|\mathcal{F}_t) = X_t$

for a filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}_+}$. Show that X is a martingale.

Problem 4. Let X and Y be martingales (with respect to the same filtration). Show that if $X_T = Y_T$ almost surely for some non-random $T > 0$, then $X_t = Y_t$ almost surely for all $0 \le t \le T$.

Problem 5. Let X be a supermartingale. Show that if $\mathbb{E}(X_t) = X_0$ for all $t \ge 0$ then X is a martingale.

Problem 6. Let the positive martingale $(Z_t)_{t\in\mathbb{T}}$ be the density process for the measure $\mathbb{Q}$ with respect to the locally equivalent measure $\mathbb{P}$. If $(M_t)_{t\in\mathbb{T}}$ is a martingale for $\mathbb{Q}$, then prove that $(M_t Z_t)_{t\in\mathbb{T}}$ is martingale for $\mathbb{P}$.

Hence show that the market $(B_t, S_t)_{t\in\mathbb{Z}_+}$ has no arbitrage if and only if there exists a positive adapted process $(\rho_t)_{t\in\mathbb{Z}_+}$ such that $(\rho_t S^i_t)_{t\in\mathbb{Z}_+}$ is a martingale for all $i = 0, 1, \ldots, d$, where $S^0 = B$.

Problem 7. (Martingale increments) Let $(M_t)_{t\in\mathbb{Z}_+}$ be a square-integrable martingale, so that $\mathbb{E}(M_t^2) < \infty$ for all $t \in \mathbb{Z}_+$. Show that the increments $M_t - M_{t-1}$ and $M_s - M_{s-1}$ are uncorrelated if $0 < s < t$. In particular, show the equality

$$\mathbb{E}(M_T^2) = M_0^2 + \mathbb{E}\left[\sum_{t=1}^{T}(M_t - M_{t-1})^2\right]$$

Problem 8. (This is a technical exercise to illustrate the fact that locally equivalent measures are not necessarily equivalent.) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with

$$\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = -1)$$

for each $n \in \mathbb{N}$ and let $S_0 = 0$ and

$$S_t = X_1 + \ldots + X_t.$$

Consider the process $(Z_t)_{t\in\mathbb{Z}_+}$ defined by

$$Z_t = \left(\frac{1-p}{p}\right)^{S_t}.$$

Show that $(Z_t)_{t\in\mathbb{Z}_+}$ is a martingale relative to its natural filtration.

Let $\mathbb{Q}$ be the measure locally equivalent to $\mathbb{P}$ defined by the density process $(Z_t)_{t\in\mathbb{Z}_+}$. Show that

$$\mathbb{Q}(X_n = 1) = 1 - p = 1 - \mathbb{Q}(X_n = -1)$$

for each $n \in \mathbb{N}$. Show that the random variables $X_1, X_2, \ldots$ are independent under $\mathbb{Q}$. Show that

$$\mathbb{P}\left(\frac{S_t}{t} \to 2p - 1\right) = 1 \text{ and } \mathbb{Q}\left(\frac{S_t}{t} \to 1 - 2p\right) = 1$$

and hence $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent if $p \neq \frac{1}{2}$.

Problem 9. (Optional sampling theorem) Let $(X_t)_{t\in\{0,\ldots,T\}}$ be a submartingale, and let σ and τ be stopping times such that $0 \leq \sigma \leq \tau \leq T$ almost surely. Show the inequality

$$\mathbb{E}(X_\sigma) \leq \mathbb{E}(X_\tau).$$

Deduce the equality $\mathbb{E}(X_\tau) = X_0$ if X is a martingale. Hint: Write

$$X_\tau - X_\sigma = \sum_{s=1}^{T} \mathbb{1}_{\{\sigma < s \leq \tau\}}(X_s - X_{s-1})$$

and apply the tower property.

Problem 10. Let $(Y_t)_{t\in\{0,\ldots,T\}}$ be a given adapted process, and let $(U_t)_{t\in\{0,\ldots,T\}}$ be its Snell envelope. Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is submartingale, then $U_t = \mathbb{E}(Y_T|\mathcal{F}_t)$.

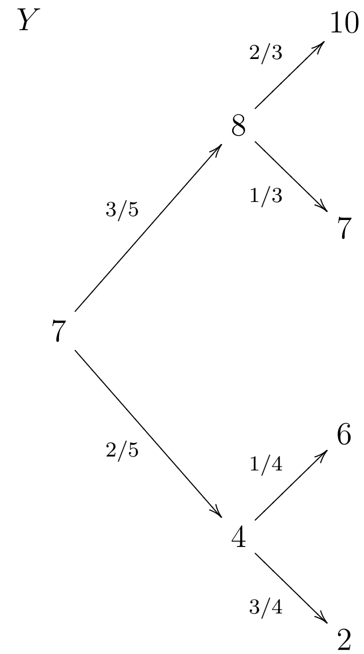
Let τ be any stopping time taking values in $\{0, \ldots, T\}$. Show that the process $(U_{t\wedge\tau})_{t\in\{0,\ldots,T\}}$ is a supermartingale.

Define the random time τ_* by

$$\tau_* = \min\left\{t \in \{0, \ldots, T\} : U_t = Y_t\right\}.$$

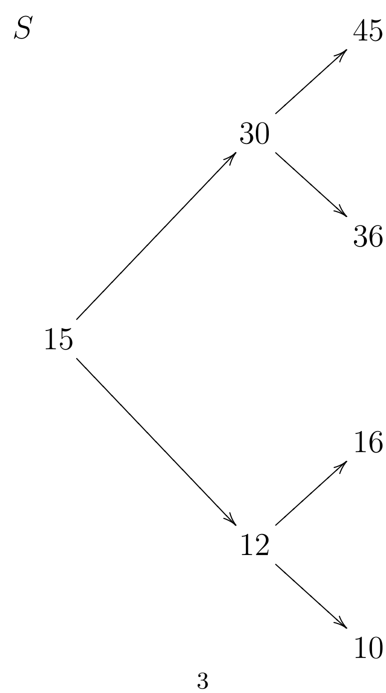
Show that τ_* is a stopping time. Furthermore, show that the process $(U_{t\wedge\tau_*})_{t\in\{0,\ldots,T\}}$ is a martingale and, in particular, $U_0 = \mathbb{E}(Y_{\tau_*})$. (That is, τ_* is an optimal stopping time, possibly different than τ^* defined in lectures.)

Problem 11. Consider the adapted process Y given by



where the above diagram should be read as $\mathbb{P}(Y_1 = 8) = 3/5$, $\mathbb{P}(Y_2 = 10|Y_1 = 8) = 2/3$, etc. Find the Snell envelope U and find the decomposition $U_t = M_t - A_t$, where M is a martingale and A is a predictable increasing process with $A_0 = 0$. Identify the two stopping times $\tau_* = \min\{t \geq 0 : U_t = Y_t\}$ and $\tau^* = \min\{t \geq 0 : A_{t+1} > 0\}$. Show directly the equality $U_0 = M_0 = \mathbb{E}(Y_{\tau_*}) = \mathbb{E}(Y_{\tau^*})$.

Problem 12. (Binomial model) Consider the following two-period market model with two assets. Let asset 0 be a riskless bank account with risk-free rate $r = 1/4$ and let asset 1 be a stock with prices given by



Find the equivalent martingale measure $\mathbb{Q}$.

Consider a European put option which strike $K = 15$ expiring at time 2. What is the no-arbitrage price of the option at time 0? What is the replicating strategy?

Now answer the same questions for an American put option with the same strike and expiration date.

Problem 13. Let $(M_t)_{t\in\mathbb{Z}_+}$ be a bounded martingale. Use Jensen's inequality to show that $(|M_t|)_{t\in\mathbb{Z}_+}$ is a submartingale.

Fix $\lambda \geq 0$ and let $\tau = \inf\{t \in \mathbb{Z}_+ : |M_t| \geq \lambda\}$. By using the optional sampling theorem at the stopping times $\tau \wedge T$ and T, show that

$$\lambda\mathbb{P}(M^* \geq \lambda) \leq \mathbb{E}(|M_T|\mathbb{1}_{\{M^*\geq\lambda\}})$$

where $M^* = \max_{t\in\{0,\dots,T\}} |M_t|$. Integrate both sides with respect to λ to show

$$\mathbb{E}[(M^*)^2] \leq 2\ \mathbb{E}(|M_T|M^*)$$

Finally, use the Cauchy-Schwarz inequality to prove

$$\mathbb{E}(\max_{t\in\{0,\dots,T\}} M_t^2) \leq 4\ \mathbb{E}(M_T^2).$$

This last inequality is called Doob's maximal inequality.

Problem 14. (Doubling strategy) Consider a market with asset 0 being cash and one risky asset with prices $(S_t)_{t\in\mathbb{Z}_+}$ having independent increments with

$$\mathbb{P}(S_{t+1} - S_t = 1) = \frac{1}{2} = \mathbb{P}(S_{t+1} - S_t = -1).$$

Let the investor's initial wealth be $X_0 = 0$ and let the strategy be

$$\pi_t = 2^{t-1}\mathbb{1}_{\{X_{t-1}\leq 0\}}.$$

Let $\tau = \inf\{t \in \mathbb{Z}_+ : X_t = 1\}$. Prove

- $\mathbb{E}(X_t) = 0$ for all $t \geq 0$.
- $\mathbb{E}(\tau) = 2$.
- $\mathbb{E}(X_\tau) = 1$. (Why does this not contradict the optional sampling theorem?)
- $\mathbb{E}(X_{\tau-1}) = -\infty$