Problem 1. Consider a one-period market model with no dividends. For the sake of this problem, call an adapted real-valued process $Z = (Z_t)_{t \in \{0,1\}}$ an ‘anti-martingale deflator’ iff

- $Z_0 \geq 0, Z_1 \geq 0$ almost surely and $\mathbb{P}(Z_0 = 0 = Z_1) < 1$,
- $Z_1 \mathbb{P}_1$ is integrable and $\mathbb{E}(Z_1 \mathbb{P}_1) = -Z_0 \mathbb{P}_0$.

Show that if there exists a numéraire portfolio, then there does not exist an anti-martingale deflator.

Problem 2. What are the economically appropriate definitions of numéraire portfolio and equivalent martingale measure in the case where the assets may pay a dividend?

Problem 3. Consider a one-period market with three assets. The first asset is a riskless asset with risk-free rate $r$. The second asset is a stock with prices $(S_t)_{t \in \{0,1\}}$. The third is a contingent claim on the stock with time $1$ price $\xi_1 = g(S_1)$, where the function $g$ is convex. Show that if there is no arbitrage, then $\xi_0 \geq \frac{1}{1+r}g[(1+r)S_0]$. Assuming $\xi_0 < \frac{1}{1+r}g[(1+r)S_0]$, find an arbitrage explicitly.

Hint: By the convexity of $g$, there exists a function $\lambda$ such that $g(x) \geq g(y) + \lambda(x)(x - y)$ for all $x, y \in \mathbb{R}$.

Problem 4. (Bayes’s formula) Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$ with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-field. Prove the identity:

$$
\mathbb{E}^\mathbb{Q}(X|\mathcal{G}) = \frac{\mathbb{E}^\mathbb{P}(ZX|\mathcal{G})}{\mathbb{E}^\mathbb{P}(Z|\mathcal{G})}
$$

for each random variable $X$ such that $X$ is $\mathbb{Q}$-integrable.

Problem 5. * Consider a trinomial two-asset model with prices $P = (B, S)$ where $B_0 = B_1 = 1$ and $S$ is given by

$$
\begin{align*}
S & \quad 3 \\
\uparrow 1/2 & \quad \downarrow 1/4 \\
2 & \quad 2 \\
\downarrow 1/4 & \quad \downarrow 1/4 \\
1 & \quad 1.
\end{align*}
$$

Find all risk-neutral measures for this model. Now introduce a call option with payout $\xi_1 = (S_1 - 2)^+$. Show that there is an open interval $I$ such that the augmented market $(B, S, \xi)$ has no arbitrage if and only if $\xi_0 \in I$.

Problem 6. Consider an arbitrage-free bond market. Let $P^T_t$ be the price of the bond of maturity $T$ at time $t$, where $1 \leq t \leq T$. Let the spot rate be $r_t = \frac{1}{P^T_t} - 1$ and the bank account be $B_t = \prod_{s=1}^{t}(1 + r_s)$ for all $t \geq 1$ as usual.

(a) Let $\mathbb{Q}$ be a risk-neutral measure, i.e. an equivalent martingale measure relative to the bank account. Show that $P^T_t = B_t \mathbb{E}^\mathbb{Q}(B_T^{-1}|\mathcal{F}_t)$ for all $0 \leq t \leq T$. 

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(b) Consider a European contingent claim with maturity $T$ and payout $r_T$. Show that this claim can be replicated by trading in bonds.

(c) Consider a forward contract initiated at time $t$ for for the payout at time $T$ of $r_T$. The forward interest rate $f^T_t$ at time $t$ for maturity $T$ is defined to be the forward price of this payout. Show that

$$f^T_t = \frac{P^{T-1}_t}{P^T_t} - 1$$

(d) Show that if the spot rate is not random, then $f^T_t = r_T$.

(e) Let $Q^T$ be a $T$-forward measure, i.e. an equivalent martingale measure relative to the bond of maturity $T$. Show that the forward rate process $(f^T_t)_{0 \leq t < T}$ is a $Q^T$ martingale.

(f) The quantity

$$y^T_t = (P^T_t)^{-\frac{1}{T-t}} - 1$$

is called the yield at time $t$ of the bond maturing at time $T$.

Show that the following are equivalent

1. $f^T_t \geq y^T_t$ a.s. for all $0 \leq t < T$
2. $T \mapsto y^T_t$ is non-decreasing a.s. for all $t \geq 0$.

(g) Show that the following are equivalent:

1. $r_t \geq 0$ a.s. for all $t \geq 1$
2. $t \mapsto B_t$ is non-decreasing a.s.
3. $T \mapsto P^T_t$ is non-increasing a.s. for each $t \geq 0$
4. $f^T_t \geq 0$ a.s. for all $0 \leq t < T$.
5. $y^T_t \geq 0$ a.s. for all $0 \leq t < T$.
6. each martingale deflator is a supermartingale.

Problem 7. (a) Let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{E}(X_n) = 1$ for all $n$. Use the Borel–Cantelli lemma to show

$$\limsup_{n \to \infty} X_n^{1/n} \leq 1 \text{ a.s.}$$

(b) Consider a bond market as in problem 6. The long rate at time $t$ is defined as $\ell_t = \lim_{T \to \infty} y^T_t$ whenever the limit exists.

Suppose that bonds are priced according to the formula in 6(a) for a fixed risk-neutral measure, and that the long rate exists a.s. at all times. Show that the long rate is non-decreasing, that is

$$\ell_s \leq \ell_t \text{ a.s. for all } 0 \leq s \leq t,$$

a fact first discovered by Dybvig, Ingersoll & Ross in 1996.

Problem 8. Let $S$ be a positive supermartingale. Show that there is a positive non-decreasing predictable process $A$ and a positive martingale $M$ such that $A_0 = M_0 = 1$ and $S_t = S_0 M_t/A_t$ for all $t \geq 0$.

Problem 9. * Let $(Y_t)_{0 \leq t \leq T}$ be a given adapted, integrable process, and let $(U_t)_{0 \leq t \leq T}$ be its Snell envelope.

(a) Show that if $Y$ is a supermartingale then $U_t = Y_t$ for all $t$, and if $Y$ is submartingale, then $U_t = \mathbb{E}(Y_T|\mathcal{F}_t)$.

(b) Let $\tau$ be any stopping time taking values in $\{0, \ldots, T\}$. Show that the process $(U_{t \land \tau})_{0 \leq t \leq T}$ is a supermartingale.
(c) Define the random time \( \tau^* \) by
\[
\tau^* = \min \{ t \in \{0, \ldots, T\} : U_t = Y_t \}.
\]
Show that \( \tau^* \) is a stopping time. Furthermore, show that the process \((U_{t \wedge \tau^*})_{t \in \{0, \ldots, T\}}\) is a martingale and, in particular, \(U_0 = \mathbb{E}(Y_{\tau^*})\). (That is, \( \tau^* \) is an optimal stopping time, possibly different than \( \tau^* \) defined in lectures.)

**Problem 10.** Let \((X_k)_{k \in K}\) be a collection of real-valued random variables, where \(K\) is an arbitrary (possibly uncountable) index set. Our aim is to show there exists a random variable \(Y\) taking values in \(\mathbb{R} \cup \{+\infty\}\) such that
- \(Y \geq X_k\) almost surely for all \(k \in K\), and
- if \(Z \geq X_k\) almost surely for all \(k \in K\) then \(Z \geq Y\) almost surely.

This will show that the \(Y = \text{ess sup}_k X_k\) exists.

(a) Show that there is no loss assuming that \(|X_k(\omega)| \leq 1\) for all \((k, \omega)\). Hint: Consider \(\tilde{X}_k = \tan^{-1}(X_k)\).

From now on, assume \(|X_k(\omega)| \leq 1\) for all \((k, \omega)\). Let \(C\) be the collection of all countable subsets of \(K\). Let
\[
x = \sup_{A \in C} \mathbb{E}[\sup_{k \in A} X_k]
\]
Let \(A_n \in C\) be such that \(\mathbb{E}[\sup_{k \in A_n} X_k] > x - 1/n\) and let \(B = \bigcup_n A_n\). Let \(Y = \sup_{k \in B} X_k\).

(b) Why is \(Y\) a random variable, i.e. measurable? Show that \(\mathbb{E}(Y) = x\).

(c) Pick a \(k \in K\), and let \(Y_k = \max\{Y, X_k\} = \sup_{h \in B \cup \{k\}} X_h\). Show that \(\mathbb{E}(Y_k) = x\). Why does this imply that \(X_k \leq Y\) almost surely?

(d) Let \(Z\) be a random variable such that \(Z \geq X_k\) a.s. for all \(k \in K\). Prove that \(Z \geq Y\) a.s.