Advanced Financial Models Michael Tehranchi
Example sheet 2 - Michaelmas 2010

Problem 1. (Bayes's formula) Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$ with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-field. Prove the identity:

$$\mathbb{E}^{\mathbb{Q}}(X|\mathcal{G}) = \frac{\mathbb{E}^{\mathbb{P}}(ZX|\mathcal{G})}{\mathbb{E}^{\mathbb{P}}(Z|\mathcal{G})}$$

for each bounded random variable X.

Solution 1. Let $Y = \frac{\mathbb{E}^{\mathbb{P}}(ZX|\mathcal{G})}{\mathbb{E}^{\mathbb{P}}(Z|\mathcal{G})}$. Note that Y is $\mathcal{G}$-measurable. Hence, we need only verify the equation $\mathbb{E}^{\mathbb{Q}}(Y\mathbb{1}_G) = \mathbb{E}^{\mathbb{Q}}(X\mathbb{1}_G)$ for all $G \in \mathcal{G}$; equivalently, we must verify $\mathbb{E}^{\mathbb{P}}(ZY\mathbb{1}_G) = \mathbb{E}^{\mathbb{P}}(ZX\mathbb{1}_G)$ for all $G \in \mathcal{G}$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}(\mathbb{1}_G YZ) &= \mathbb{E}^{\mathbb{P}}[\mathbb{E}(\mathbb{1}_G YZ|\mathcal{G})] \text{ tower property} \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_G Y \mathbb{E}(Z|\mathcal{G})] \text{ taking out what's known} \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_G \mathbb{E}^{\mathbb{P}}(XZ|\mathcal{G})] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}(\mathbb{1}_G XZ|\mathcal{G})] \text{ pulling in what's known} \\
&= \mathbb{E}^{\mathbb{P}}(\mathbb{1}_G XZ) \text{ tower property}
\end{aligned}$$

Problem 2. Prove the tower property of conditional expectations.

Solution 2. Let X be integrable random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ be sigma-fields. Let $Y = \mathbb{E}(X|\mathcal{G})$ and $Z = \mathbb{E}(X|\mathcal{H})$.

First we show $\mathbb{E}(Y|\mathcal{H}) = Z$. Notice that for every $H \in \mathcal{H}$ we have $\mathbb{E}(Z\mathbb{1}_H) = \mathbb{E}(X\mathbb{1}_H)$ by the definition of conditional expectation, and $\mathbb{E}(X\mathbb{1}_H) = \mathbb{E}(Y\mathbb{1}_H)$, again by the definition of conditional expectation since $H \in \mathcal{G}$. It follows that $\mathbb{E}(Z\mathbb{1}_H) = \mathbb{E}(Y\mathbb{1}_H)$ for all $H \in \mathcal{H}$, proving the claim.

Now we show $\mathbb{E}(Z|\mathcal{G}) = Z$. First note that Z is $\mathcal{G}$-measurable, since $\{Z \leq z\} \in \mathcal{G}$ and Z is $\mathcal{H}$-measurable by definition. Since (trivially) $\mathbb{E}(Z\mathbb{1}_G) = \mathbb{E}(Z\mathbb{1}_G)$ for each $G \in \mathcal{G}$, we have $\mathbb{E}(Z|\mathcal{G}) = Z$ as claimed.

Problem 3. Suppose $X = (X_t)_{t\in\mathbb{Z}_+}$ has the following properties: For all $t \in \mathbb{Z}_+$

- $\mathbb{E}(|X_t|) < \infty$, and
- $\mathbb{E}(X_{t+1}|\mathcal{F}_t) = X_t$

for a filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}_+}$. Show that X is a martingale.

Solution 3. We must verify the equality $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $0 \leq s \leq t$. The proof is by induction. First, by assumption $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ when $t = s + 1$. Now if $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for some $t > s$ then

$$\begin{aligned}
\mathbb{E}(X_{t+1}|\mathcal{F}_s) &= \mathbb{E}[\mathbb{E}(X_{t+1}|\mathcal{F}_t)|\mathcal{F}_s] \\
&= \mathbb{E}(X_t|\mathcal{F}_s) = X_s
\end{aligned}$$

by the tower property. Hence $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $t, s \in \mathbb{Z}_+$ with $t > s$.

Problem 4. Let X and Y be martingales (with respect to the same filtration). Show that if $X_T = Y_T$ almost surely for some fixed $T > 0$, then $X_t = Y_t$ almost surely for all $0 \leq t \leq T$.

Solution 4. If $X_T = Y_T$ almost surely, then

$$X_t = \mathbb{E}(X_T|\mathcal{F}_t) = \mathbb{E}(Y_T|\mathcal{F}_t) = Y_t$$

for all $0 \le t \le T$.

Problem 5. Let X be a supermartingale. Show that if $\mathbb{E}(X_t) = X_0$ for all $t \ge 0$ then X is a martingale.

Solution 5. If the index set $\mathbb{T}$ is discrete, we can use the Doob decomposition $X = M - A$, where M is a martingale and A is a non-decreasing process with $A_0 = 0$. Hence

$$X_0 = \mathbb{E}(X_t) = \mathbb{E}(M_t - A_t) = M_0 - \mathbb{E}(A_t)$$

Since $M_0 = X_0$, we have $\mathbb{E}(A_t) = 0$ for all $t \ge 0$. But $A_t \ge 0$ almost surely, and we must conclude that $A_t = 0$ almost surely.

But for a general index set $\mathbb{T}$ we can argue by bare hands. Let $Y_{s,t} = X_s - \mathbb{E}(X_t|\mathcal{F}_s)$ for each $0 \le s \le t$. The assumption implies $\mathbb{E}(Y_{s,t}) = 0$ since

$$\mathbb{E}(Y_{s,t}) = \mathbb{E}(X_s) - \mathbb{E}[\mathbb{E}(X_t|\mathcal{F}_s)] = X_0 - \mathbb{E}(X_t)$$

by the tower property. But $Y_{s,t} \ge 0$ almost surely since X is a supermartingale, and we can conclude that $Y_{s,t} = 0$ almost surely as desired.

Problem 6. Let the positive martingale $(Z_t)_{t\in\mathbb{T}}$ be the density process for the measure $\mathbb{Q}$ with respect to the locally equivalent measure $\mathbb{P}$. If $(M_t)_{t\in\mathbb{T}}$ is a martingale for $\mathbb{Q}$, then prove that $(M_t Z_t)_{t\in\mathbb{T}}$ is martingale for $\mathbb{P}$.

Hence show that the market $(B_t, S_t)_{t\in\mathbb{Z}_+}$ has no arbitrage if and only if there exists a positive adapted process $(\rho_t)_{t\in\mathbb{Z}_+}$ such that $(\rho_t S_t^{(i)})_{t\in\mathbb{Z}_+}$ is a martingale for all $i = 0, 1, \ldots, d$.

Solution 6. First a technical note: if X is $\mathcal{G}$-measurable for some sigma-field $\mathcal{G} \subseteq \mathcal{F}$, we may replace the measure $\mathbb{P}$ by its restriction $\mathbb{P}|_{\mathcal{G}}$ to $\mathcal{G}$ when computing expectations and conditional expectations of X. For instance $\mathbb{E}^{\mathbb{P}}(X) = \mathbb{E}^{\mathbb{P}|_{\mathcal{G}}}(X)$.

Applying Bayes's formula yields

$$\begin{aligned} M_s &= \mathbb{E}^{\mathbb{Q}}(M_t|\mathcal{F}_s) \\ &= \mathbb{E}^{\mathbb{Q}_t}(M_t|\mathcal{F}_s) \\ &= \frac{\mathbb{E}^{\mathbb{P}_t}(Z_t M_t|\mathcal{F}_s)}{\mathbb{E}^{\mathbb{P}_t}(Z_t|\mathcal{F}_s)} \\ &= \frac{\mathbb{E}^{\mathbb{P}}(Z_t M_t|\mathcal{F}_s)}{\mathbb{E}^{\mathbb{P}}(Z_t|\mathcal{F}_s)} \end{aligned}$$

where we have made use of the fact that both M_t and Z_t are $\mathcal{F}_t$-measurable. Since $\mathbb{E}^{\mathbb{P}}(Z_t|\mathcal{F}_s) = Z_s$ we have $\mathbb{E}^{\mathbb{P}}(Z_t M_t|\mathcal{F}_s) = Z_s M_s$ as desired.

Now, the first fundamental theorem of asset pricing says that the existence of a locally equivalent measure $\mathbb{Q}$ such that $\tilde{S} = S/B$ is a martingale. If Z be the density process of $\mathbb{Q}$, then S/B is a $\mathbb{Q}$-martingale if and only if ZS/B is a $\mathbb{P}$-martingale. That is, we need only choose $\rho_t = Z_t/B_t$.

Problem 7. (Martingale increments) Let $(M_t)_{t\in\mathbb{Z}_+}$ be a square-integrable martingale, so that $\mathbb{E}(M_t^2) < \infty$ for all $t \in \mathbb{Z}_+$. Show that the increments $M_t - M_{t-1}$ and $M_s - M_{s-1}$ are uncorrelated if $0 < s < t$. In particular, show the equality

$$\mathbb{E}(M_T^2) = M_0^2 + \mathbb{E}\left[\sum_{t=1}^{T}(M_t - M_{t-1})^2\right].$$

Solution 7. Note martingale increments are of mean zero:

$$\mathbb{E}(M_t - M_{t-1}) = M_0 - M_0 = 0$$

Similarly, for $0 < s < t$ we have

$$\begin{aligned}\mathbb{E}[(M_t - M_{t-1})(M_s - M_{s-1})] &= \mathbb{E}\{\mathbb{E}[(M_t - M_{t-1})(M_s - M_{s-1})\big|\mathcal{F}_s]\} \\ &= \mathbb{E}[\mathbb{E}(M_t - M_{t-1}\big|\mathcal{F}_s)(M_s - M_{s-1})] \\ &= \mathbb{E}[(M_s - M_s)(M_s - M_{s-1})] = 0\end{aligned}$$

Hence $\mathrm{Cov}(M_t - M_{t-1}, M_s - M_{s-1}) = 0$.

Therefore, M_t can be written as a sum of orthogonal terms as

$$M_T = M_0 + \sum_{s=1}^{T}(M_s - M_{s-1}).$$

The Pythagorean theorem proves the identity.

Problem 8. (This is a technical exercise to illustrate the fact that locally equivalent measures are not necessarily equivalent.) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with

$$\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = -1)$$

for each $n \in \mathbb{N}$ and let $S_0 = 0$ and

$$S_t = X_1 + \ldots + X_t.$$

Consider the process $(Z_t)_{t\in\mathbb{Z}_+}$ defined by

$$Z_t = \left(\frac{1-p}{p}\right)^{S_t}.$$

Show that $(Z_t)_{t\in\mathbb{Z}_+}$ is a martingale relative to its natural filtration.

Let $\mathbb{Q}$ be the measure locally equivalent to $\mathbb{P}$ defined by the density process $(Z_t)_{t\in\mathbb{Z}_+}$. Show that

$$\mathbb{Q}(X_n = 1) = 1 - p = 1 - \mathbb{Q}(X_n = -1)$$

for each $n \in \mathbb{N}$. Show that the random variables $X_1, X_2, \ldots$ are independent under $\mathbb{Q}$. Show that

$$\mathbb{P}\left(\frac{S_t}{t} \to 2p - 1\right) = 1 \text{ and } \mathbb{Q}\left(\frac{S_t}{t} \to 1 - 2p\right) = 1$$

and hence $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent if $p \neq \frac{1}{2}$.

Solution 8. Notice that the natural filtration of S coincides with the natural filtration of Z, as $s \mapsto \left(\frac{1-p}{p}\right)^s$ is continuous and invertible. Let's denote this common filtration by $(\mathcal{F}_t)_{t\in\mathbb{Z}_+}$ in the following.

Let's prove that Z is a $\mathbb{P}$-martingale: First, note that $Z_t \leq (\max\{p/(1-p),(1-p)/p\})^t$, so that $\mathbb{E}^{\mathbb{P}}|Z_t| < \infty$. Now,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big(Z_{t+1}\big|\mathcal{F}_t\big) &= Z_t\mathbb{E}^{\mathbb{P}}\left[\big(\frac{1-p}{p}\big)^{X_{t+1}}\big|\mathcal{F}_t\right] \\
&= Z_t\mathbb{E}^{\mathbb{P}}\left[\big(\frac{1-p}{p}\big)^{X_{t+1}}\right] \\
&= Z_t\Big(p\frac{1-p}{p}+(1-p)\frac{p}{1-p}\Big) \\
&= Z_t.
\end{aligned}$$

We now show that under $\mathbb{Q}$ the X_i's are independent with $\mathbb{Q}(X_i=1)=1-p$: Fix $n\in\mathbb{N}$ and $\{\alpha_i\}_{i=1}^n \in \{-1,1\}^n$. Let $n_+ = \#\{i : \alpha_i = 1\}$ and $n_- = \#\{i : \alpha_i = -1\}$ so that $n_+ + n_- = n$. Indeed, we have two computations:

$$\begin{aligned}
\mathbb{Q}\left(\bigcap_{i=1}^n\{X_i=\alpha_i\}\right) &= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{1}_{\bigcap_{i=1}^n\{X_i=\alpha_i\}}Z_n\Big) \\
&= \Big(\frac{1-p}{p}\Big)^{n_+-n_-}p^{n_+}(1-p)^{n_-} \\
&= (1-p)^{n_+}p^{n_-}
\end{aligned}$$

and

$$\prod_{i=1}^n \mathbb{Q}(X_i=\alpha_i) = (1-p)^{n_+}p^{n_-}.$$

Finally, as the X_i's are independent, and integrable under both $\mathbb{P}$ and $\mathbb{Q}$. Therefore the strong law of large numbers applies and we conclude that:

$$\frac{S_t}{t} \to 2p-1 \quad \mathbb{P}-\text{a.s.}$$

and

$$\frac{S_t}{t} \to 1-2p \quad \mathbb{Q}-\text{a.s.}$$

Problem 9. (Optional sampling theorem) Let $(X_t)_{t\in\{0,\ldots,T\}}$ be a submartingale, and let σ and τ be stopping times such that $0\leq\sigma\leq\tau\leq T$ almost surely. Show the inequality

$$\mathbb{E}(X_\sigma) \leq \mathbb{E}(X_\tau).$$

Deduce the equality $\mathbb{E}(X_\tau) = X_0$ if X is a martingale. Hint: Write

$$X_\tau - X_\sigma = \sum_{s=1}^T \mathbb{1}_{\{\sigma<s\leq\tau\}}(X_s - X_{s-1})$$

and apply the tower property.

Solution 9. Following the hint, we write:

$$\begin{aligned}\mathbb{E}(X_\tau - X_\sigma) &= \sum_{s=1}^{T} \mathbb{E}(\mathbb{1}_{\{\sigma < s \le \tau\}}(X_s - X_{s-1})) \\ &= \sum_{s=1}^{T} \mathbb{E}[\mathbb{E}(\mathbb{1}_{\{\sigma < s \le \tau\}}(X_s - X_{s-1}) \big| F_{s-1})]\end{aligned}$$

Notice that $\{\sigma < s\} = \{\sigma \le s-1\} \in \mathcal{F}_{s-1}$ and $\{\tau \ge s\} = \{\tau \le s-1\}^c \in \mathcal{F}_{s-1}$. Therefore, $\mathbb{1}_{\{\sigma < s \le \tau\}}$ is $\mathcal{F}_{s-1}$ measurable, and we can take it out of the expectation with respect to $\mathcal{F}_{s-1}$:

$$\mathbb{E}(X_\tau - X_\sigma) = \sum_{s=1}^{T} \mathbb{E}[\mathbb{1}_{\{\sigma < s \le \tau\}} \mathbb{E}((X_s - X_{s-1}) \big| F_{s-1})]$$

This is now clear that this is non-negative as X is a submartingale.

The second statement follows from the fact if X is a martingale then both X and $-X$ are submartingales and that $\sigma = 0$ is a stopping time.

Problem 10. Let $(Y_t)_{t \in \{0,\dots,T\}}$ be a given adapted process, and let $(U_t)_{t \in \{0,\dots,T\}}$ be its Snell envelope. Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is submartingale, then $U_t = \mathbb{E}(Y_T | \mathcal{F}_t)$.

Let τ be any stopping time taking values in $\{0, \dots, T\}$. Show that the process $(U_{t\wedge\tau})_{t \in \{0,\dots,T\}}$ is a supermartingale.

Define the random time τ_* by

$$\tau_* = \min\{t \in \{0, \dots, T\} : U_t = Y_t\}.$$

Show that τ_* is a stopping time. Furthermore, show that the process $(U_{t\wedge\tau_*})_{t \in \{0,\dots,T\}}$ is a martingale and, in particular, $U_0 = \mathbb{E}(Y_{\tau_*})$. (That is, τ_* is an optimal stopping time, possibly different than τ^* defined in lectures.)

Solution 10. In both cases, we proceed by induction. First suppose that Y is a supermartingale, and that $U_{t+1} = Y_{t+1}$ for some $t < T$. Then

$$U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} = \max\{Y_t, \mathbb{E}(Y_{t+1}|\mathcal{F}_t)\} = Y_t$$

since $Y_t \ge \mathbb{E}(Y_{t+1}|\mathcal{F}_t)$ by assumption, completing the induction. Similarly, suppose that Y is a submartingale, and that $U_{t+1} = \mathbb{E}(Y_T|\mathcal{F}_{t+1})$. Then

$$U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} = \max\{Y_t, \mathbb{E}[\mathbb{E}(Y_T|\mathcal{F}_{t+1})|\mathcal{F}_t]\} = \mathbb{E}(Y_T|\mathcal{F}_t)$$

by the tower property and the assumption $Y_t \le \mathbb{E}(Y_T|\mathcal{F}_t)$, and we're done.

Since U is a supermaringale and the event $\{\tau \ge t+1\} = \{\tau \le t\}^c$ is in $\mathcal{F}_t$, we have

$$\begin{aligned}\mathbb{E}[U_{(t+1)\wedge\tau} - U_{t\wedge\tau}|\mathcal{F}_t] &= \mathbb{E}[\mathbb{1}_{\{t+1 \le \tau\}}(U_{t+1} - U_t)|\mathcal{F}_t] \\ &= \mathbb{1}_{\{t+1 \le \tau\}} \mathbb{E}[U_{t+1} - U_t|\mathcal{F}_t] \\ &\le 0\end{aligned}$$

so the stopped process $(U_{t\wedge\tau})_{t \in \{0,\dots,T\}}$ is also supermartingale.

Now, the event

$$\{\tau_* > t\} = \{Y_0 < U_0, \dots, Y_t < U_t\}$$

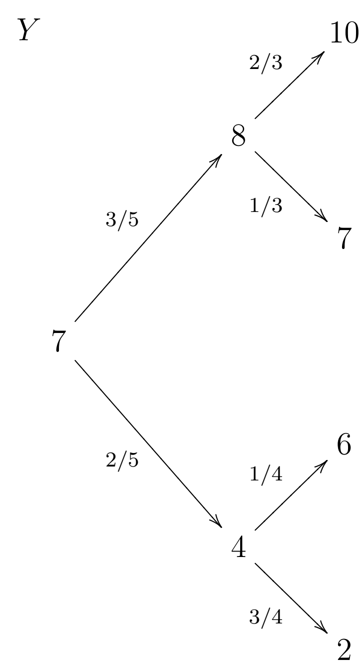
is $\mathcal{F}_t$-measurable since both Y and U are adapted, hence τ_* is a stopping time.

Since $U_t = \mathbb{E}(U_{t+1}|\mathcal{F}_t)$ on the event $\{t+1 \leq \tau_*\}$ we have

$$\begin{aligned} U_{(t+1)\wedge\tau_*} - U_{t\wedge\tau_*} &= \mathbb{1}_{\{t+1\leq\tau_*\}}(U_{t+1} - U_t) \\ &= \mathbb{1}_{\{t+1\leq\tau_*\}}[U_{t+1} - \mathbb{E}(U_{t+1}|\mathcal{F}_t)]. \end{aligned}$$

and the result follows upon taking conditional expectations.

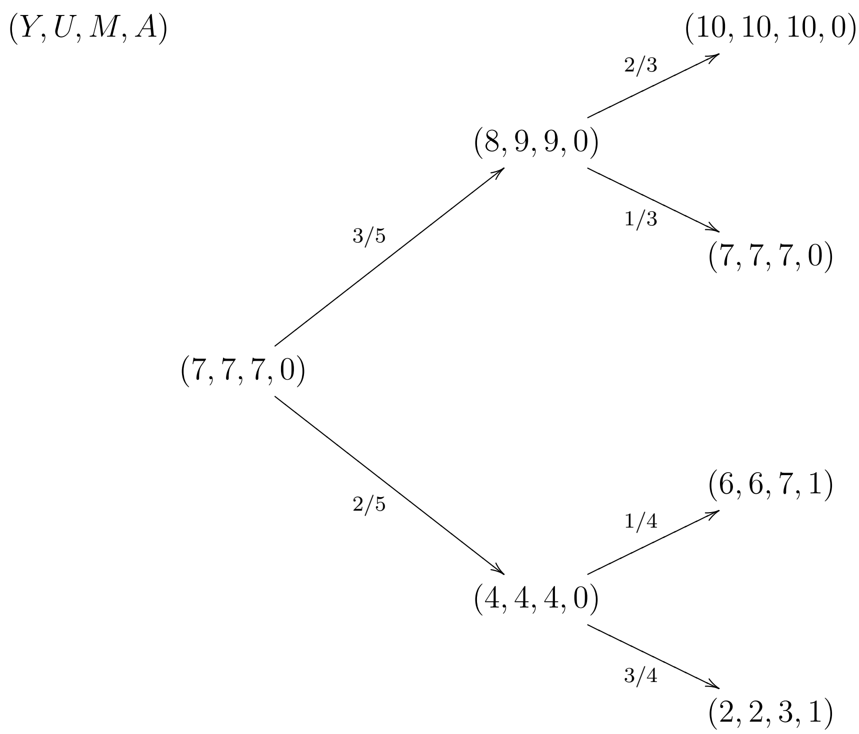
Problem 11. Consider the adapted process Y given by



where the above diagram should be read as $\mathbb{P}(Y_1 = 8) = 3/5$, $\mathbb{P}(Y_2 = 10|Y_1 = 8) = 2/3$, etc. Find the Snell envelope U and find the decomposition $U_t = M_t - A_t$, where M is a martingale and A is a predictable increasing process with $A_0 = 0$. Identify the two stopping times $\tau_* = \min\{t \geq 0 : U_t = Y_t\}$ and $\tau^* = \min\{t \geq 0 : A_{t+1} > 0\}$. Show directly the equality $U_0 = M_0 = \mathbb{E}(Y_{\tau_*}) = \mathbb{E}(Y_{\tau^*})$.

Solution 11. Let $(\mathcal{F}_t)_{t\in\{0,1,2\}}$ be the filtration generated by the process Y. The Snell envelope U is defined by $U_2 = Y_2$ and $U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)$. The Doob decomposition can be found by $M_0 = U_0$ and $M_{t+1} = M_t + U_{t+1} - \mathbb{E}(U_{t+1}|\mathcal{F}_t)$ and $A_{t+1} = A_t + U_t - \mathbb{E}(U_{t+1}|\mathcal{F}_t)$.

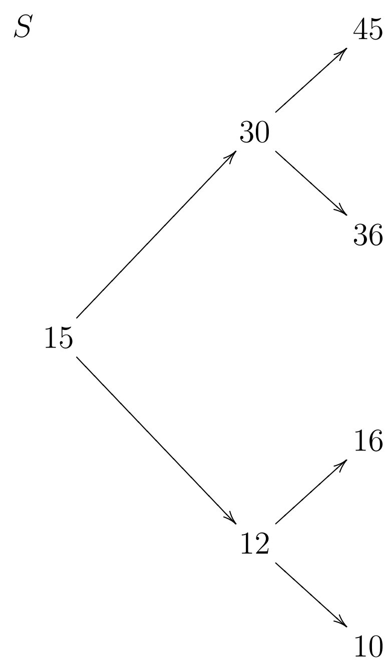
The values found this way can be read from the diagram



Then $\tau_* = 0$ almost surely, while

$$\tau^*(\omega) = \begin{cases} 2 & \text{if } \omega \in \{Y_1 = 8\} \\ 1 & \text{if } \omega \in \{Y_1 = 4\} \end{cases}$$

Problem 12. (Binomial model) Consider the following two-period market model with two assets. Let asset 0 be a riskless bank account with risk-free rate $r = 1/4$ and let asset 1 be a stock with prices given by



Find the equivalent martingale measure $\mathbb{Q}$.

Consider a European put option which strike $K = 15$ expiring at time 2. What is the no-arbitrage price of the option at time 0? What is the replicating strategy?

Now answer the same questions for an American put option with the same strike and expiration date.

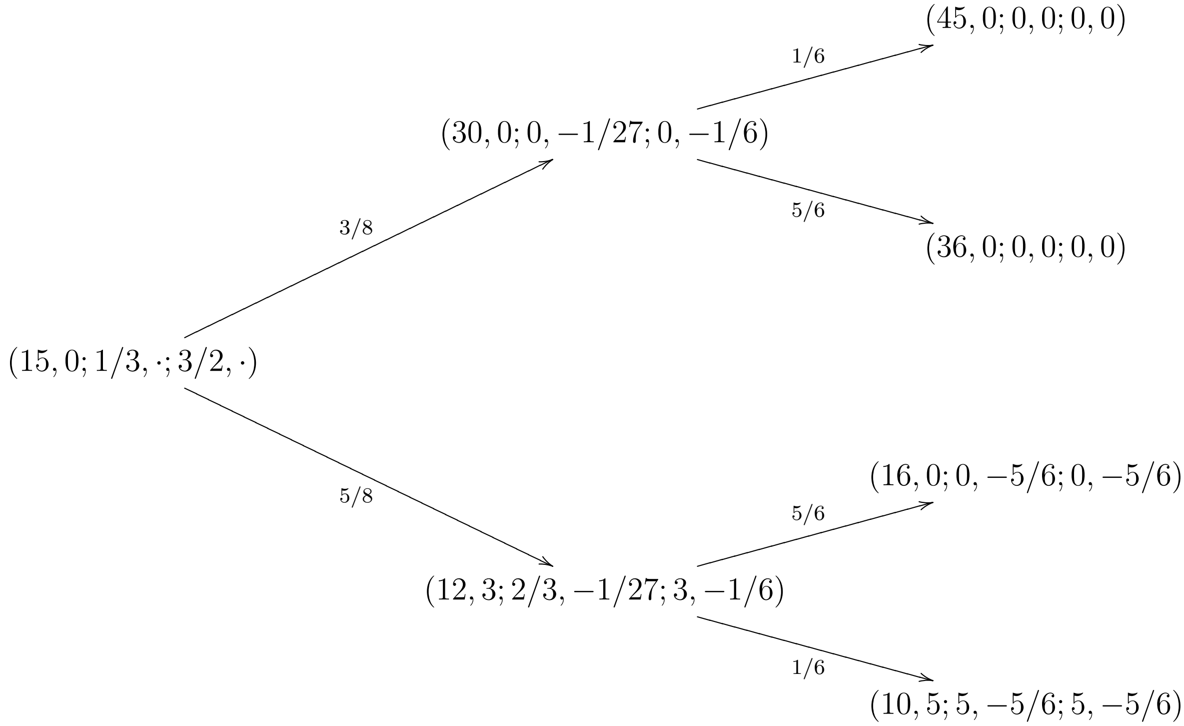
Solution 12. The equivalent martingale measure must satisfy $\mathbb{E}^{\mathbb{Q}}\big((1+r)^{-1}S_{t+1}|\mathcal{F}_t\big) = S_t$. For instance solving $45q + 36(1-q) = (1+1/4)(30)$ yields $q = 1/6 = \mathbb{Q}(S_2 = 45|S_1 = 30)$.

Since the market is complete, the there is a unique no arbitrage price P^E which can be described by the backward recursion $P^E_T = (K - S_T)^+$ and by the formula $P^E_t = \mathbb{E}^{\mathbb{Q}}\big((1+r)^{-1}P^E_{t+1}|\mathcal{F}_t\big)$. The replicating strategy is found by solving $P^E_t(1+r) + \pi^E_{t+1}[S_{t+1} - S_t(1+r)] = P^E_{t+1}$.

For the American option, the price P^A is given inductively by $P^A_T = (K - S_T)^+$ and $P^A_t = \max\{(K - S_t)^+, \mathbb{E}^{\mathbb{Q}}\big((1+r)^{-1}P^A_{t+1}|\mathcal{F}_t\big)\}$. The super-replicating strategy is found by solving $P^A_t(1+r) + \pi^A_{t+1}[S_{t+1} - S_t(1+r)] = P^A_{t+1}$ on $\{t < \tau^*\}$ where τ^* is an optimal stopping time.

The results can be read from the diagram, where $\xi_t = (K - S_t)^+$.

$(S, \xi; P^E, \pi^E; P^A, \pi^A)$



Notice that for any optimal stopping time τ^*, we have $\tau^* = 1$ on $\{S_1 = 12\}$. If the holder of the option fails to exercise optimally, the seller of the option need only to hedge the remaining one-period European option (in the bottom right corner of the diagram) at a cost of $2/3$ and pocket the rest $3 - 2/3 = 7/3$.

Problem 13. Let $(M_t)_{t\in\mathbb{Z}_+}$ be a bounded martingale. Use Jensen's inequality to show that $(|M_t|)_{t\in\mathbb{Z}_+}$ is a submartingale.

Fix $\lambda \geq 0$ and let $\tau = \inf\{t \in \mathbb{Z}_+ : |M_t| \geq \lambda\}$. By using the optional sampling theorem at the stopping times $\tau \wedge T$ and T, show that

$$\lambda \mathbb{P}(M^* \geq \lambda) \leq \mathbb{E}(|M_T| \mathbb{1}_{\{M^* \geq \lambda\}})$$

where $M^* = \max_{t\in\{0,\ldots,T\}} |M_t|$. Integrate both sides with respect to λ to show

$$\mathbb{E}[(M^*)^2] \leq 2\, \mathbb{E}(|M_T| M^*)$$

Finally, use the Cauchy-Schwarz inequality to prove

$$\mathbb{E}(\max_{t\in\{0,\ldots,T\}} M_t^2) \leq 4\, \mathbb{E}(M_T^2).$$

This last inequality is called Doob's maximal inequality.

Solution 13. By the conditional version of Jensen's inequality $\mathbb{E}(|M_{t+1}| \,\big|\mathcal{F}_t) \geq |\mathbb{E}(M_{t+1}|\mathcal{F}_t)| = |M_t|$ and hence $(|M_t|)_{t\in\{t\in\mathbb{Z}_+\}}$ is a submartingale.

Since $\tau \wedge T \leq T$ and $(|M_t|)_{t\in\{0,\ldots,T\}}$ is a submartingale, we have by the optional sampling theorem

$$\mathbb{E}(|M_{\tau\wedge T}|) \leq \mathbb{E}(|M_T|).$$

We have $|M_{\tau\wedge T}| \geq \lambda \mathbb{1}_{\{\tau \leq T\}} + |M_T|\mathbb{1}_{\{\tau > T\}}$ so that

$$\begin{aligned} \lambda\, \mathbb{P}(\tau \leq T) &\leq \mathbb{E}(|M_{\tau\wedge T}|) - \mathbb{E}(|M_T|\mathbb{1}_{\{\tau > T\}}) \\ &\leq \mathbb{E}(|M_T|) - \mathbb{E}(|M_T|\mathbb{1}_{\{\tau > T\}}) = \mathbb{E}(|M_T|\mathbb{1}_{\{\tau > T\}}). \end{aligned}$$

The result now follows from noticing $\{\tau \leq T\} = \{M^* \geq \lambda\}$.

Integrate both sides of the above inequality with respect to λ. On the left side we have

$$\int_0^\infty \lambda \mathbb{P}(M^* \geq \lambda) d\lambda = \mathbb{E}\left(\int_0^\infty \lambda \mathbb{1}_{\{M^* \geq \lambda\}} d\lambda \right) = \frac{1}{2}\mathbb{E}[(M^*)^2]$$

and on the right side

$$\int_0^\infty \mathbb{E}(M_T \mathbb{1}_{\{M^* \geq \lambda\}}) d\lambda = \mathbb{E}\left(M_T \int_0^\infty \mathbb{1}_{\{M^* \geq \lambda\}} d\lambda \right) = \mathbb{E}(M_T M^*).$$

(The interchange of expectation and integration with respect to λ is justified in both cases since the integrands are almost surely non-negative.)

$$\mathbb{E}[(M^*)^2] \leq 2\mathbb{E}(|M_T|M^*) \leq 2\left(\mathbb{E}[M_T^2]\right)^{1/2}\left(\mathbb{E}[(M^*)^2]\right)^{1/2}$$

by the Cauchy-Schwarz inequality. Squaring both sides and dividing by $\mathbb{E}[(M^*)^2]$ yields the desired conclusion.

Problem 14. (Doubling strategy) Consider a market with asset 0 being cash and one risky asset with prices $(S_t)_{t\in\mathbb{Z}_+}$ having independent increments with

$$\mathbb{P}(S_{t+1} - S_t = 1) = \frac{1}{2} = \mathbb{P}(S_{t+1} - S_t = -1).$$

Let the investor's initial wealth be $X_0 = 0$ and let the strategy be

$$\pi_t = 2^{t-1}\mathbb{1}_{\{X_{t-1} \leq 0\}}.$$

Let $\tau = \inf\{t \in \mathbb{Z}_+ : X_t = 1\}$. Prove

- $\mathbb{E}(X_t) = 0$ for all $t \geq 0$.
- $\mathbb{E}(\tau) = 2$.
- $\mathbb{E}(X_\tau) = 1$. (Why does this not contradict the optional sampling theorem?)
- $\mathbb{E}(X_{\tau-1}) = -\infty$

Solution 14. The process X is a martingale since S is martingale, and for each $t > 0$, the strategy $(\pi_s)_{s\in\{1,\ldots,t\}}$ is bounded by 2^{t-1}.

The random variable $\tau = \inf\{t \geq 1 : S_t - S_{t-1} = 1\}$ is geometric with success parameter $p = 1/2$; that is, $\mathbb{P}(\tau = k) = 2^{-k}$ for $k = 1, 2, \ldots$ and hence $\mathbb{E}(\tau) = 1/p = 2$.

In particular, $\mathbb{P}(\tau < \infty) = 1$ so that X_τ is well-defined and equals 1 almost surely. This is not a contradiction of the optional sampling theorem as τ is not bounded.

Since

$$X_t = \begin{cases} -2^t + 1 & \text{for } t = 0, \ldots, \tau - 1 \\ 1 & \text{for } t \geq \tau \end{cases}$$

we have

$$\mathbb{E}(X_{\tau-1}) = \sum_{k=1}^\infty \mathbb{E}(X_{k-1}|\tau = k)\mathbb{P}(\tau = k) = \sum_{k=1}^\infty (-2^{k-1} + 1)2^{-k} = -\infty.$$