Problem 1. Consider a one-period market model with no dividends. For the sake of this problem, call an adapted real-valued process $Z = (Z_t)_{t \in \{0,1\}}$ an ‘anti-martingale deflator’ iff

- $Z_0 \geq 0$, $Z_1 \geq 0$ almost surely and $\mathbb{P}(Z_0 = 0 = Z_1) < 1$,
- $Z_1 P_1$ is integrable and $\mathbb{E}(Z_1 P_1) = -Z_0 P_0$

Show that if there exists a numéraire portfolio, then there does not exist an anti-martingale deflator.

Solution 1. Suppose there exists a numéraire portfolio $\eta$ and suppose $Z_t \geq 0$ for $t = 0, 1$ and $\mathbb{E}(Z_1 P_1) = -Z_0 P_0$. Multiplying by $\eta$ yields

(*) $-Z_0 N_0 = \mathbb{E}(Z_1 N_1)$

where $N_t = \eta \cdot P_t$ for $t = 0, 1$. Since $N_t > 0$ a.s. we have $Z_t N_t \geq 0$ for $t = 0, 1$. Combined with equation (*) we have $Z_0 N_0 = 0$. Since $N_0 \neq 0$ we conclude that $Z_0 = 0$. Equation (*) also says $\mathbb{E}(Z_1 N_1) = 0$, so by the pigeonhole principle $Z_1 N_1 = 0$ a.s. Again, since $N_1 \neq 0$ a.s. we have $Z_1 = 0$ a.s. In particular, $Z_0, Z_1$ is not an anti-martingale deflator.

Problem 2. What are the economically appropriate definitions of numéraire portfolio and equivalent martingale measure in the case where the assets may pay a dividend?

Solution 2. A numéraire is a previsible process $\eta$ such that $\eta_t \cdot P_0 > 0$ and $\eta_{t+1} \cdot P_t = \eta_t \cdot (P_t + \delta_t) > 0$ a.s. for all $t \geq 1$. An equivalent martingale measure relative to the numéraire is a measure $\mathbb{Q}$ under which

$$\frac{P_t}{N_t} = \mathbb{E}_\mathbb{Q}\left(\frac{P_{t+1} + \delta_{t+1}}{N_{t+1}} | \mathcal{F}_t\right)$$

for all $t \geq 0$. It remains to explain why these definitions are economically appropriate. First note that assuming there is a numéraire according to our revised definition, implies that there is an arbitrage if and only if there is a terminal consumption arbitrage. Next, there is an equivalent martingale measure according to our revised definition if and only if there is a martingale deflator.

Problem 3. Consider a one-period market with three assets. The first asset is a riskless asset with risk-free rate $r$. The second asset is a stock with prices $(S_t)_{t \in \{0,1\}}$. The third is a contingent claim on the stock with time 1 price $\xi_t = g(S_t)$, where the function $g$ is convex. Show that if there is no arbitrage, then $\xi_0 \geq \frac{1}{1+r} g[(1+r)S_0]$. Assuming $\xi_0 < \frac{1}{1+r} g[(1+r)S_0]$, find an arbitrage explicitly.

Hint: By the convexity of $g$, there exists a function $\lambda$ such that $g(x) \geq g(y) + \lambda(x)(x - y)$ for all $x, y \in \mathbb{R}$.

Solution 3. Suppose $\xi_0 < \frac{1}{1+r} g[(1+r)S_0]$ and let $L = \lambda[(1+r)S_0]$. Consider the portfolio $H = (-\frac{1}{1+r} g[(1+r)S_0] + LS_0, -L, +1)$. Note

$$c_0 = -H \cdot (B_0, S_0, \xi_0)$$

$$= \frac{1}{1+r} g[(1+r)S_0] - \xi_0 > 0$$
and
\[ c_1 = H \cdot (B_1, S_1, \xi_1) = -g((1 + r)S_0 + LS_0(1 + r) - LS_1 + g(S_1) \geq 0 \]

**Problem 4.** (Baye’s formula) Let \( P \) and \( Q \) be equivalent probability measures defined on \((\Omega, \mathcal{F})\) with density \( Z = \frac{dQ}{dP} \). Let \( \mathcal{G} \subseteq \mathcal{F} \) be a sigma-field. Prove the identity:
\[
E^Q(X|\mathcal{G}) = \frac{E^P(ZX|\mathcal{G})}{E^P(Z|\mathcal{G})}
\]
for each random variable \( X \) such that \( X \) is \( Q \)-integrable.

**Solution 4.** Let \( Y = \frac{E^P(ZX|\mathcal{G})}{E^P(Z|\mathcal{G})} \). Note that \( Y \) is \( \mathcal{G} \)-measurable. Hence, we need only verify the equation \( E^Q(Y \cdot 1_G) = E^Q(X \cdot 1_G) \) for all \( G \in \mathcal{G} \); equivalently, we must verify \( E^P(ZY \cdot 1_G) = E^P(ZX \cdot 1_G) \) for all \( G \in \mathcal{G} \).

\[
E^P(1_G Y Z) = E^P[E^P([1_G Y Z]|\mathcal{G})] \text{ tower property}
\]
\[
= E^P[1_G Y E^P(Z|\mathcal{G})] \text{ taking out what’s known}
\]
\[
= E^P[1_G E^P(XZ|\mathcal{G})] \text{ pulling in what’s known}
\]
\[
= E^P([1_G XZ]|\mathcal{G}) \text{ tower property}
\]

**Problem 5.** * Consider a trinomial two-asset model with prices \( P = (B, S) \) where \( B_0 = B_1 = 1 \) and \( S \) is given by

\[
\begin{array}{c c c c}
 & & & 3 \\
 & 2 & & 2 \\
& & 1/4 & 1/4
\end{array}
\]

Find all risk-neutral measures for this model. Now introduce a call option with payout \( \xi_1 = (S_1 - 2)^+ \). Show that there is an open interval \( I \) such that the augmented market \((B, S, \xi)\) has no arbitrage if and only if \( \xi_0 \in I \).

**Solution 5.** A risk neutral measure solves \( E^Q(S_1) = S_0 \), (no discounting is needed since the numéraire is cash)

\[
3p + 2q + r = 2
\]
\[
p + q + r = 1
\]

and hence \((p, q, r) = (p, 1 - 2p, p)\) for \( 0 < p < 1/2 \), where \( p = \mathbb{Q}\{S_1 = 3\} \), etc. By the fundamental theorem of asset pricing, there is no arbitrage if and only if \( \xi_0 = E^Q(\xi_1) = p \).

So \( I = (0, 1/2) \).

**Problem 6.** Consider an arbitrage-free bond market. Let \( P^T_t \) be the price of the bond of maturity \( T \) at time \( t \), where \( 1 \leq t \leq T \). Let the spot rate be \( r_t = \frac{1}{P^T_{t-1}} - 1 \) and the bank account be \( B_t = \prod_{s=1}^{t}(1 + r_s) \) for all \( t \geq 1 \) as usual.
(a) Let \( Q \) be a risk-neutral measure, i.e. an equivalent martingale measure relative to the bank account. Show that

\[
P_t^T = B_t \mathbb{E}^Q(B_T^{-1}|F_t)
\]

for all \( 0 \leq t \leq T \).

(b) Consider a European contingent claim with maturity \( T \) and payout \( r_T \). Show that this claim can be replicated by trading in bonds.

(c) Consider a forward contract initiated at time \( t \) for for the payout at time \( T \) of \( r_T \). The forward interest rate \( f_t^T \) at time \( t \) for maturity \( T \) is defined to be the forward price of this payout. Show that

\[
f_t^T = \frac{P_t^T - 1}{P_t^T} - 1
\]

(d) Show that if the spot rate is not random, then \( f_t^T = r_T \).

(e) Let \( Q^T \) be a \( T \)-forward measure, i.e. an equivalent martingale measure relative to the bond of maturity \( T \). Show that the forward rate process \( (f_t^T)_{0 \leq t \leq T} \) is a \( Q^T \) martingale.

(f) The quantity

\[
y_t^T = (P_t^T)^{-\frac{1}{T-t}} - 1
\]

is called the yield at time \( t \) of the bond maturing at time \( T \).

Show that the following are equivalent

1. \( f_t^T \geq y_t^T \) a.s. for all \( 0 \leq t < T \)
2. \( T \mapsto y_t^T \) is non-decreasing a.s. for all \( t \geq 0 \).

(g) Show that the following are equivalent:

1. \( r_t \geq 0 \) a.s. for all \( t \geq 1 \)
2. \( t \mapsto B_t \) is non-decreasing a.s.
3. \( T \mapsto P_t^T \) is non-increasing a.s. for each \( t \geq 0 \)
4. \( f_t^T \geq 0 \) a.s. for all \( 0 \leq t < T \).
5. \( y_t^T \geq 0 \) a.s. for all \( 0 \leq t < T \).
6. each martingale deflator is a supermartingale.

**Solution 6.**

(a) The discounted price \( P_t^T/B = (P_t^T/B_t)_{0 \leq t \leq T} \) is a martingale for any risk-neutral measure. The result follows from \( P_t^T = 1 \) and the martingale property.

(b) Work backwards: Note that the payout \( r_T \) is \( \mathcal{F}_{T-1} \) measurable, and can be realised by holding \( r_T \) bonds of maturity \( T \) during the period \( (T-1, T] \). The time \( T-1 \) cost of this strategy is \( r_T P_{T-1}^T = 1 - P_T^T P_{T-1}^T \). Now to replicate this time \( T-1 \) payout by holding one bond of maturity \( T-1 \) and selling one bond of maturity \( T \).

In summary, the replication strategy is at time \( 0 \) to buy one bond of maturity \( T-1 \) and to sell one bond of maturity \( T \). At time \( T-1 \), invest the payout of the bond of maturity \( T-1 \) into \( 1/P_T^T \) bonds of maturity \( T \).

(c) A **dual approach:** From lectures

\[
f_t^T = \mathbb{E}^{Q^T}(r_T|F_t)
\]
where \( Q^T \) is a \( T \)-forward measure. Since \( r_T = \frac{1}{P_{T-1}^T} - 1 \) and by changing to a risk-neutral measure (using part (a)) we have

\[
P_t^T \mathbb{E}_Q^T \left( \frac{1}{P_{T-1}^T} \mid F_t \right) = B_t \mathbb{E}_Q^Q \left( \frac{B_{T-1}^r}{\mathbb{E}_Q(B_{T-1}^1 \mid F_{T-1}) B_{T-1}} \right) \mid F_t \text{ cond on } F_{T-1} \text{ and tower}
\]

\[
= B_t \mathbb{E}_Q^Q(B_{T-1}^1 \mid F_t)
\]

\[
= P_t^{T-1}
\]

A primal approach: Consider the forward claim initiated at time \( t \) with time \( T \) payout \( \xi_T = r_T - f_t^T \). From part (b), the cost at time \( t \) to replicate \( r_T \) is \( P_t^{T-1} - P_t^T \), and the cost at time \( t \) to replicate the \( F_t \)-measurable payout \( f_t^T \) is \( f_t^T P_t^T \). Hence \( \xi_t = P_t^{T-1} - (1 + f_t^T) P_t^T \). But the initial price of a forward is \( \xi_t = 0 \). Solving for \( f_t^T \) yields the formula.

(d) If the spot rate is not random, then from part (a) we have \( P_t^T = B_t/B_T \), and hence

\[
f_t^T = \frac{P_t^{T-1}}{P_t^T} - 1 = \frac{B_T}{B_{T-1}} - 1 = r_T
\]

(e) This follows from Doob’s observation that \( M_t = \mathbb{E}(\xi \mid F_t) \) defines a martingale whenever \( \xi \) is integrable.

(f) \( f_t^T \geq y_t^T \iff (P_t^{T-1})^{T-t} \geq (P_t^T)^{T-t-1} \iff y_t^{T-1} \leq y_t^T \)

(g) (1) \( \iff B_t = (1 + r_t)B_{t-1} \geq B_{t-1} \iff (2)

(2) \( \iff P_t^{T+1} = B_t \mathbb{E}_Q^Q(B_{T+1}^1 \mid F_t) \leq B_t \mathbb{E}_Q^Q(B_T^1 \mid F_t) = P_t^T \iff (3).

(3) \( \iff 1 = P_t^T \geq P_t^{T+1} = B_t \mathbb{E}_Q^Q(B_{T+1}^1 \mid F_t) = B_t/B_{t+1} \) since \( B \) is predictable \( \iff (2).

(4) \( \iff P_t^{T-1} \geq P_t^T \iff (3).

(5) \( \iff P_t^{T+1} \leq 1 \iff r_{t+1} \geq 0 \iff (1)

(3) \( \iff P_t^T \leq 1 \iff (5)

(6) \( \iff r_t = \frac{Y_t}{\mathbb{E}(Y_t \mid F_{t-1})} - 1 \geq 0 \iff (1)

Problem 7. (a) Let \( X_1, X_2, \ldots \) be a sequence of non-negative random variables such that \( \mathbb{E}(X_n) = 1 \) for all \( n \). Use the Borel–Cantelli lemma to show

\[
\limsup_{n \to \infty} X_n^{1/n} \leq 1 \text{ a.s.}
\]

(b) Consider a bond market as in problem 6. The long rate at time \( t \) is defined as \( \ell_t = \lim_{T \to \infty} y_t^T \) whenever the limit exists.

Suppose that bonds are priced according to the formula in 6(a) for a fixed risk-neutral measure, and that the long rate exists a.s. at all times. Show that the long rate is non-decreasing, that is

\[
\ell_s \leq \ell_t \text{ a.s. for all } 0 \leq s \leq t,
\]

a fact first discovered by Dybvig, Ingersoll & Ross in 1996.

Solution 7. (a) For each \( \epsilon > 0 \) we have

\[
\sum_{n=1}^{\infty} \mathbb{P}(X_n > (1 + \epsilon)^n) \leq \sum_{n=1}^{\infty} (1 + \epsilon)^{-n} = \frac{1}{\epsilon} < \infty
\]

by Markov’s inequality. The first Borel–Cantelli lemma then says

\[
\mathbb{P}(X_n^{1/n} > 1 + \epsilon \text{ infinitely often}) = 0
\]
This shows \( \limsup_{n \to \infty} X_n^{1/n} \leq 1 \) as claimed.
(b) Now, let \( P_t^T \) be the bond price, \( B_t \) the bank account, and \( \tilde{P}_t^T = P_t^T / B_t \) the discounted bond price. Suppose the long rate \( \ell_t \) exists, so that
\[
\ell_t = \lim_{T \to \infty} (P_t^T)^{-1/(T-t)} - 1 = \lim_{n \to \infty} (\tilde{P}_t^n)^{-1/n} - 1
\]
By assumption, the discounted bond prices are given by
\[
\tilde{P}_t^T = \mathbb{E}^Q(B_T^1 | \mathcal{F}_t)
\]
each \( 0 \leq t \leq T \) and a fixed risk-neutral measure \( Q \), and, in particular, \( \tilde{P}^T \) is a martingale for each \( T > 0 \).
Fix \( 0 \leq s \leq t \), and let
\[
X_n = \frac{\tilde{P}_t^n}{\tilde{P}_s^n}.
\]
Note \( \mathbb{E}(X_n) = \mathbb{E}[\mathbb{E}(X_n | \mathcal{F}_s)] = 1 \) for each \( n \). The first part implies
\[
\frac{\ell_s + 1}{\ell_t + 1} = \lim_{n \to \infty} (X_n)^{1/n} \leq 1 \text{ a.s.}
\]
as required.

Problem 8. Let \( S \) be a positive supermartingale. Show that there is a positive non-decreasing predictable process \( A \) and a positive martingale \( M \) such that \( A_0 = M_0 = 1 \) and \( S_t = S_0 M_t / A_t \) for all \( t \geq 0 \).

Solution 8. Let
\[
A_t = \prod_{s=1}^{t} \frac{S_{s-1}}{\mathbb{E}(S_s | \mathcal{F}_{s-1})}
\]
so that \( A \) is predictable and non-decreasing since \( A_{t+1} = A_t S_t / \mathbb{E}(S_{t+1} | \mathcal{F}_t) \geq A_t \) since \( S \) is a supermartingale.
Let
\[
M_t = \prod_{s=1}^{t} \frac{S_s}{\mathbb{E}(S_s | \mathcal{F}_{s-1})}
\]
Apply Problem 3 from example sheet 1 to show that \( M \) is a martingale. By construction \( M / A = S / S_0 \).

Problem 9. * Let \( (Y_t)_{0 \leq t \leq T} \) be a given adapted, integrable process, and let \( (U_t)_{0 \leq t \leq T} \) be its Snell envelope.
(a) Show that if \( Y \) is a supermartingale then \( U_t = Y_t \) for all \( t \), and if \( Y \) is submartingale, then \( U_t = \mathbb{E}(Y_T | \mathcal{F}_t) \).
(b) Let \( \tau \) be any stopping time taking values in \( \{0, \ldots, T\} \). Show that the process \( (U_{t \wedge \tau})_{0 \leq t \leq T} \) is a supermartingale.
(c) Define the random time \( \tau_* \) by
\[
\tau_* = \min \{ t \in \{0, \ldots, T\} : U_t = Y_t \}.
\]
Show that \( \tau_* \) is a stopping time. Furthermore, show that the process \( (U_{t \wedge \tau_*})_{t \in \{0, \ldots, T\}} \) is a martingale and, in particular, \( U_0 = \mathbb{E}(Y_{\tau_*}) \). (That is, \( \tau_* \) is an optimal stopping time, possibly different than \( \tau^* \) defined in lectures.)
Problem 10. Let \((X_k)_{k \in K}\) be a collection of real-valued random variables, where \(K\) is an arbitrary (possibly uncountable) index set. Our aim is to show there exists a random variable \(Y\) taking values in \(\mathbb{R}\) such that \(Y \geq X_k\) almost surely for all \(k \in K\) and \(Z \geq Y\) almost surely.

This will show that \(Y = \text{ess sup}_k X_k\) exists.

(a) Show that there is no loss assuming that \(|X_k(\omega)| \leq 1\) for all \((k, \omega)\). Hint: Consider \(\hat{X}_k = \tan^{-1}(X_k)\).

From now on, assume \(|X_k(\omega)| \leq 1\) for all \((k, \omega)\). Let \(C\) be the collection of all countable subsets of \(K\). Let

\[
x = \sup_{A \in C} \mathbb{E}[\sup_{k \in A} X_k]
\]

Let \(A_n \in C\) be such that \(\mathbb{E}[\sup_{k \in A_n} X_k] > x - 1/n\) and let \(B = \bigcup_n A_n\). Let \(Y = \sup_{k \in B} X_k\).

(b) Why is \(Y\) a random variable, i.e. measurable? Show that \(\mathbb{E}(Y) = x\).

(c) Pick a \(k \in K\), and let \(Y_k = \max\{Y, X_k\} = \sup_{h \in B \cup \{k\}} X_h\). Show that \(\mathbb{E}(Y_k) = x\). Why does this imply that \(X_k \leq Y\) almost surely?

(d) Let \(Z\) be a random variable such that \(Z \geq X_k\) a.s. for all \(k \in K\). Prove that \(Z \geq Y\) a.s.

Solution 10. (a) Suppose we know that every family of uniformly bounded random variables (indexed by some arbitrary index set) has an essential supremum. So given the family \((X_k)_{k \in K}\) let \(\hat{X}_k = \tan^{-1}(X_k)\). Then the \(\hat{X}_k\) are bounded so there exists \(\hat{Y} = \text{ess sup}_k \hat{X}_k\).
Let \( Y = \tan \hat{Y} \). Since the function \( \tan \) is strictly increasing it is easy to see that \( Y \) has the properties characterising the essential supremum of \( (X_k)_k \).

(b) Since \( A_n \) is countable, the function \( Y_n = \sup_{k \in C_n} X_k \) is measurable for each \( n \). Also \( \bigcup_n A_n = B \) is countable and hence \( Y \) is also a random variable. For instance, note

\[
\{ Y > b \} = \bigcup_{k \in B} \{ X_k > b \}.
\]

Now, by replacing \( A_n \) with \( \bigcup_{i=1}^n A_i \) we may assume \( A_{n-1} \subseteq A_n \) for all \( n \geq 1 \), and hence we have \( Y = \sup_n Y_n = \lim_n Y_n \). The result follows from the bounded convergence theorem.

(c) Since \( Y_k \geq Y \) we have \( \mathbb{E}(Y_k) \geq \mathbb{E}(Y) = x \). On the other hand,

\[
x = \sup_{A \in \mathcal{C}} \mathbb{E}[\sup_{h \in A} X_h] \\
\geq \mathbb{E}[\sup_{h \in B \cup \{k\}} X_h] \\
= \mathbb{E}(Y_k).
\]

Since \( Y - Y_k \geq 0 \) a.s. and \( \mathbb{E}(Y - Y_k) = 0 \), the pigeonhole principle implies \( Y = Y_k \) a.s. Since \( \max\{Y, X_k\} = Y \) we have \( X_k \leq Y \).

(d) If \( \mathbb{P}(Z \geq X_k) = 1 \) for all \( k \in K \), then

\[
\mathbb{P}(Z \geq Y) = \mathbb{P}(\cap_{k \in B} \{ Z \geq \hat{X}_k \}) = 1
\]

since \( B \) is countable.