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Example sheet 2 - Michaelmas 2019

Problem 1. Consider a one-period market model with no dividends. For the sake of this problem, call an adapted real-valued process $Z = (Z_t)_{t\in\{0,1\}}$ an 'anti-martingale deflator' iff

- $Z_0 \geq 0$, $Z_1 \geq 0$ almost surely and $\mathbb{P}(Z_0 = 0 = Z_1) < 1$,
- $Z_1 P_1$ is integrable and $\mathbb{E}(Z_1 P_1) = -Z_0 P_0$

Show that if there exists a numéraire portfolio, then there does not exist an anti-martingale deflator.

Solution 1. Suppose there exists an numéraire portfolio η and suppose $Z_t \geq 0$ for $t = 0, 1$ and $\mathbb{E}(Z_1 P_1) = -Z_0 P_0$. Multiplying by η yields

$$(*) \qquad -Z_0 N_0 = \mathbb{E}(Z_1 N_1)$$

where $N_t = \eta \cdot P_t$ for $t = 0, 1$. Since $N_t > 0$ a.s. we have $Z_t N_t \geq 0$ for $t = 0, 1$. Combined with equation $(*)$ we have $Z_0 N_0 = 0$. Since $N_0 \neq 0$ we conclude that $Z_0 = 0$. Equation $(*)$ also says $\mathbb{E}(Z_1 N_1) = 0$, so by the pigeonhole principle $Z_1 N_1 = 0$ a.s. Again, since $N_1 \neq 0$ a.s. we have $Z_1 = 0$ a.s. In particular, Z_0, Z_1 is not an anti-martingale deflator.

Problem 2. What are the economically appropriate definitions of numéraire portfolio and equivalent martingale measure in the case where the assets may pay a dividend?

Solution 2. A numéraire is a previsible process η such that $\eta_1 \cdot P_0 > 0$ and $\eta_{t+1} \cdot P_t = \eta_t \cdot (P_t + \delta_t) > 0$ a.s. for all $t \geq 1$. An equivalent martingale measure relative to the numéraire is a measure $\mathbb{Q}$ under which

$$\frac{P_t}{N_t} = \mathbb{E}^{\mathbb{Q}}\left(\frac{P_{t+1} + \delta_{t+1}}{N_t} | \mathcal{F}_t\right)$$

for all $t \geq 0$. It remains to explain why these definitions are economically appropriate. First note that assuming there is a numéraire according to our revised definition, implies that there is an arbitrage if and only if there is a terminal consumption arbitrage. Next, there is an equivalent martingale measure according to our revised definition if and only if there is a martingale deflator.

Problem 3. Consider a one-period market with three assets. The first asset is a riskless asset with risk-free rate r. The second asset is a stock with prices $(S_t)_{t\in\{0,1\}}$. The third is a contingent claim on the stock with time 1 price $\xi_1 = g(S_1)$, where the function g is convex. Show that if there is no arbitrage, then $\xi_0 \geq \frac{1}{1+r} g[(1+r)S_0]$. Assuming $\xi_0 < \frac{1}{1+r} g[(1+r)S_0]$, find an arbitrage explicitly.

Hint: By the convexity of g, there exists a function λ such that $g(x) \geq g(y) + \lambda(x)(x - y)$ for all $x, y \in \mathbb{R}$.

Solution 3. Suppose $\xi_0 < \frac{1}{1+r} g[(1+r)S_0]$ and let $L = \lambda[(1+r)S_0]$. Consider the portfolio $H = (-\frac{1}{1+r} g[(1+r)S_0] + LS_0, -L, +1)$. Note

$$\begin{aligned} c_0 &= -H \cdot (B_0, S_0, \xi_0) \\ &= \frac{1}{1+r} g[(1+r)S_0] - \xi_0 > 0 \end{aligned}$$

and

$$\begin{aligned} c_1 &= H \cdot (B_1, S_1, \xi_1) \\ &= -g[(1+r)S_0] + LS_0(1+r) - LS_1 + g(S_1) \geq 0 \end{aligned}$$

Problem 4. (Bayes's formula) Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$ with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-field. Prove the identity:

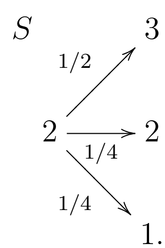
$$\mathbb{E}^{\mathbb{Q}}(X|\mathcal{G}) = \frac{\mathbb{E}^{\mathbb{P}}(ZX|\mathcal{G})}{\mathbb{E}^{\mathbb{P}}(Z|\mathcal{G})}$$

for each random variable X such that X is $\mathbb{Q}$-integrable.

Solution 4. Let $Y = \frac{\mathbb{E}^{\mathbb{P}}(ZX|\mathcal{G})}{\mathbb{E}^{\mathbb{P}}(Z|\mathcal{G})}$. Note that Y is $\mathcal{G}$-measurable. Hence, we need only verify the equation $\mathbb{E}^{\mathbb{Q}}(Y\mathbb{1}_G) = \mathbb{E}^{\mathbb{Q}}(X\mathbb{1}_G)$ for all $G \in \mathcal{G}$; equivalently, we must verify $\mathbb{E}^{\mathbb{P}}(ZY\mathbb{1}_G) = \mathbb{E}^{\mathbb{P}}(ZX\mathbb{1}_G)$ for all $G \in \mathcal{G}$.

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}(\mathbb{1}_G YZ) =& \mathbb{E}^{\mathbb{P}}[\mathbb{E}(\mathbb{1}_G YZ|\mathcal{G})] \text{ tower property} \\ =& \mathbb{E}^{\mathbb{P}}[\mathbb{1}_G Y \mathbb{E}(Z|\mathcal{G})] \text{ taking out what's known} \\ =& \mathbb{E}^{\mathbb{P}}[\mathbb{1}_G \mathbb{E}^{\mathbb{P}}(XZ|\mathcal{G})] \\ =& \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}(\mathbb{1}_G XZ|\mathcal{G})] \text{ pulling in what's known} \\ =& \mathbb{E}^{\mathbb{P}}(\mathbb{1}_G XZ) \text{ tower property} \end{aligned}$$

Problem 5. * Consider a trinomial two-asset model with prices $P = (B, S)$ where $B_0 = B_1 = 1$ and S is given by



Find all risk-neutral measures for this model. Now introduce a call option with payout $\xi_1 = (S_1 - 2)^+$. Show that there is an open interval I such that the augmented market (B, S, ξ) has no arbitrage if and only if $\xi_0 \in I$.

Solution 5. A risk neutral measure solves $\mathbb{E}^{\mathbb{Q}}(S_1) = S_0$, (no discounting is needed since the numéraire is cash)

$$\begin{aligned} 3p + 2q + r &= 2 \\ p + q + r &= 1 \end{aligned}$$

and hence $(p, q, r) = (p, 1-2p, p)$ for $0 < p < 1/2$, where $p = \mathbb{Q}\{S_1 = 3\}$, etc. By the fundamental theorem of asset pricing, there is no arbitrage if and only if $\xi_0 = \mathbb{E}^{\mathbb{Q}}(\xi_1) = p$. So $I = (0, 1/2)$.

Problem 6. Consider an arbitrage-free bond market. Let P_t^T be the price of the bond of maturity T at time t, where $1 \leq t \leq T$. Let the spot rate be $r_t = \frac{1}{P_{t-1}^t} - 1$ and the bank account be $B_t = \prod_{s=1}^t (1 + r_s)$ for all $t \geq 1$ as usual.

(a) Let $\mathbb{Q}$ be a risk-neutral measure, i.e. an equivalent martingale measure relative to the bank account. Show that

$$P_t^T = B_t \, \mathbb{E}^{\mathbb{Q}}(B_T^{-1}|\mathcal{F}_t)$$

for all $0 \leq t \leq T$.
(b) Consider a European contingent claim with maturity T and payout r_T. Show that this claim can be replicated by trading in bonds.
(c) Consider a forward contract initiated at time t for for the payout at time T of r_T. The forward interest rate f_t^T at time t for maturity T is defined to be the forward price of this payout. Show that

$$f_t^T = \frac{P_t^{T-1}}{P_t^T} - 1$$

(d) Show that if the spot rate is not random, then $f_t^T = r_T$.
(e) Let $\mathbb{Q}^T$ be a T-forward measure, i.e. an equivalent martingale measure relative to the bond of maturity T. Show that the forward rate process $(f_t^T)_{0 \leq t < T}$ is a $\mathbb{Q}^T$ martingale.
(f) The quantity

$$y_t^T = (P_t^T)^{-\frac{1}{T-t}} - 1$$

is called the yield at time t of the bond maturing at time T.

Show that the following are equivalent

(1) $f_t^T \geq y_t^T$ a.s. for all $0 \leq t < T$
(2) $T \mapsto y_t^T$ is non-decreasing a.s. for all $t \geq 0$.

(g) Show that the following are equivalent:

(1) $r_t \geq 0$ a.s. for all $t \geq 1$
(2) $t \mapsto B_t$ is non-decreasing a.s.
(3) $T \mapsto P_t^T$ is non-increasing a.s. for each $t \geq 0$
(4) $f_t^T \geq 0$ a.s for all $0 \leq t < T$.
(5) $y_t^T \geq 0$ a.s. for all $0 \leq t < T$.
(6) each martingale deflator is a supermartingale.

Solution 6. (a) The discounted price $P^T/B = (P_t^T/B_t)_{0 \leq t \leq T}$ is a martingale for any risk-neutral measure. The result follows from $P_T^T = 1$ and the martingale property.
(b) Work backwards: Note that the payout r_T is $\mathcal{F}_{T-1}$ measurable, and can be realised by holding r_T bonds of maturity T during the period $(T-1, T]$. The time $T-1$ cost of this strategy is $r_T P_{T-1}^T = 1 - P_{T-1}^T$. Now to replicate this time $T-1$ payout by holding one bond of maturity $T-1$ and selling one bond of maturity T.

In summary, the replication strategy is at time 0 to buy one bond of maturity $T-1$ and to sell one bond of maturity T. At time $T-1$, invest the payout of the bond of maturity $T-1$ into $1/P_{T-1}^T$ bonds of maturity T.
(c) *A dual approach:* From lectures

$$f_t^T = \mathbb{E}^{\mathbb{Q}^T}(r_T|\mathcal{F}_t)$$

where $\mathbb{Q}^T$ is a T-forward measure. Since $r_T = \frac{1}{P^T_{T-1}} - 1$ and by changing to a risk-neutral measure (using part (a)) we have

$$\begin{aligned} P^T_t \mathbb{E}^{\mathbb{Q}^T}\left(\frac{1}{P^T_{T-1}}|\mathcal{F}_t\right) &= B_t \mathbb{E}^{\mathbb{Q}}\left(\frac{B_T^{-1}}{\mathbb{E}^{\mathbb{Q}}(B_T^{-1}|\mathcal{F}_{T-1})B_{T-1}}|\mathcal{F}_t\right) \text{ cond on } \mathcal{F}_{T-1} \text{ and tower} \\ &= B_t \mathbb{E}^{\mathbb{Q}}(B_{T-1}^{-1}|\mathcal{F}_t) \\ &= P^{T-1}_t \end{aligned}$$

A primal approach: Consider the forward claim initiated at time t with time T payout $\xi_T = r_T - f^T_t$. From part (b), the cost at time t to replicate r_T is $P^{T-1}_t - P^T_t$, and the cost at time t to replicate the $\mathcal{F}_t$-measurable payout f^T_t is $f^T_t P^T_t$. Hence $\xi_t = P^{T-1}_t - (1 + f^T_t)P^T_t$. But the initial price of a forward is $\xi_t = 0$. Solving for f^T_t yields the formula.
(d) If the spot rate is not random, then from part (a) we have $P^T_t = B_t/B_T$, and hence

$$f^T_t = \frac{P^{T-1}_t}{P^T_t} - 1 = \frac{B_T}{B_{T-1}} - 1 = r_T$$

(e) This follows from Doob's observation that $M_t = \mathbb{E}(\xi|\mathcal{F}_t)$ defines a martingale whenever ξ is integrable.
(f) $f^T_t \ge y^T_t \Leftrightarrow (P^{T-1}_t)^{T-t} \ge (P^T_t)^{T-t-1} \Leftrightarrow y^{T-1}_t \le y^T_t$
(g) (1) $\Leftrightarrow B_t = (1 + r_t)B_{t-1} \ge B_{t-1} \Leftrightarrow$ (2)
(2) $\Rightarrow P^{T+1}_t = B_t \mathbb{E}^{\mathbb{Q}}(B^{-1}_{T+1}|\mathcal{F}_t) \le B_t \mathbb{E}^{\mathbb{Q}}(B^{-1}_T|\mathcal{F}_t) = P^T_t \Leftrightarrow$ (3).
(3) $\Rightarrow 1 = P^t_t \ge P^{t+1}_t = B_t \mathbb{E}^{\mathbb{Q}}(B^{-1}_{t+1}|\mathcal{F}_t) = B_t/B_{t+1}$ since B is predictable $\Leftrightarrow$ (2).
(4) $\Leftrightarrow P^{T-1}_t \ge P^T_t \Leftrightarrow$ (3).
(5) $\Rightarrow P^{t+1}_t \le 1 \Leftrightarrow r_{t+1} \ge 0 \Leftrightarrow$ (1)
(3) $\Rightarrow P^T_t \le 1 \Leftrightarrow$ (5)
(6) $\Leftrightarrow r_t = \frac{Y_{t-1}}{\mathbb{E}(Y_t|\mathcal{F}_{t-1})} - 1 \ge 0 \Leftrightarrow$ (1)

Problem 7. (a) Let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{E}(X_n) = 1$ for all n. Use the Borel–Cantelli lemma to show

$$\limsup_{n\to\infty} X_n^{1/n} \le 1 \text{ a.s.}$$

(b) Consider a bond market as in problem 6. The *long rate* at time t is defined as $\ell_t = \lim_{T\to\infty} y^T_t$ whenever the limit exists.

Suppose that bonds are priced according to the formula in 6(a) for a fixed risk-neutral measure, and that the long rate exists a.s. at all times. Show that the long rate is non-decreasing, that is

$$\ell_s \le \ell_t \text{ a.s. for all } 0 \le s \le t,$$

a fact first discovered by Dybvig, Ingersoll & Ross in 1996.

Solution 7. (a) For each $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n > (1+\epsilon)^n] \le \sum_{n=1}^{\infty} (1+\epsilon)^{-n} = \frac{1}{\epsilon} < \infty$$

by Markov's inequality. The first Borel–Cantelli lemma then says

$$\mathbb{P}(X_n^{1/n} > 1 + \epsilon \text{ infinitely often}) = 0$$

This shows $\limsup_{n\to\infty} X_n^{1/n} \le 1$ as claimed.
(b)Now, let P_t^T be the bond price, B_t the bank account, and $\tilde{P}_t^T = P_t^T/B_t$ the discounted bond price. Suppose the long rate ℓ_t exists, so that

$$\ell_t = \lim_{T\to\infty} (P_t^T)^{-1/(T-t)} - 1 = \lim_{n\to\infty} (\tilde{P}_t^n)^{-1/n} - 1$$

By assumption, the discounted bond prices are given by

$$\tilde{P}_t^T = \mathbb{E}^{\mathbb{Q}}(B_T^{-1}|\mathcal{F}_t)$$

each $0 \le t \le T$ and a fixed risk-neutral measure $\mathbb{Q}$, and, in particular, $\tilde{P}^T$ is a martingale for each $T > 0$.

Fix $0 \le s \le t$, and let

$$X_n = \frac{\tilde{P}_t^n}{\tilde{P}_s^n}.$$

Note $\mathbb{E}(X_n) = \mathbb{E}[\mathbb{E}(X_n|\mathcal{F}_s)] = 1$ for each n. The first part implies

$$\frac{\ell_s + 1}{\ell_t + 1} = \lim_{n\to\infty} (X_n)^{1/n} \le 1 \text{ a.s.}$$

as required.

Problem 8. Let S be a positive supermartingale. Show that there is a positive non-decreasing predictable process A and a positive martingale M such that $A_0 = M_0 = 1$ and $S_t = S_0 M_t/A_t$ for all $t \ge 0$.

Solution 8. Let

$$A_t = \prod_{s=1}^{t} \frac{S_{s-1}}{\mathbb{E}(S_s|\mathcal{F}_{s-1})}$$

so that A is predictable and non-decreasing since $A_{t+1} = A_t S_t/\mathbb{E}(S_{t+1}|\mathcal{F}_t) \ge A_t$ since S is a supermartingale.

Let

$$M_t = \prod_{s=1}^{t} \frac{S_s}{\mathbb{E}(S_s|\mathcal{F}_{s-1})}$$

Apply Problem 3 from example sheet 1 to show that M is a martingale. By construction $M/A = S/S_0$.

Problem 9. * Let $(Y_t)_{0\le t\le T}$ be a given adapted, integrable process, and let $(U_t)_{0\le t\le T}$ be its Snell envelope.
(a) Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is submartingale, then $U_t = \mathbb{E}(Y_T|\mathcal{F}_t)$.
(b) Let τ be any stopping time taking values in $\{0, \dots, T\}$. Show that the process $(U_{t\wedge\tau})_{0\le t\le T}$ is a supermartingale.
(c) Define the random time τ_* by

$$\tau_* = \min\{t \in \{0, \dots, T\} : U_t = Y_t\}.$$

Show that τ_* is a stopping time. Furthermore, show that the process $(U_{t\wedge\tau_*})_{t\in\{0,\dots,T\}}$ is a martingale and, in particular, $U_0 = \mathbb{E}(Y_{\tau_*})$. (That is, τ_* is an optimal stopping time, possibly different than τ^* defined in lectures.)

Solution 9. (a) In both cases we proceed by induction. First suppose that Y is a supermartingale, and that $U_{t+1} = Y_{t+1}$ for some $t < T$. Then

$$U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} = \max\{Y_t, \mathbb{E}(Y_{t+1}|\mathcal{F}_t)\} = Y_t$$

since $Y_t \geq \mathbb{E}(Y_{t+1}|\mathcal{F}_t)$ by assumption, completing the induction. Similarly, suppose that Y is a submartingale, and that $U_{t+1} = \mathbb{E}(Y_T|\mathcal{F}_{t+1})$. Then

$$U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} = \max\{Y_t, \mathbb{E}[\mathbb{E}(Y_T|\mathcal{F}_{t+1})|\mathcal{F}_t]\} = \mathbb{E}(Y_T|\mathcal{F}_t)$$

by the tower property and the assumption $Y_t \leq \mathbb{E}(Y_T|\mathcal{F}_t)$, and we're done.
(b) Since U is a supermaringale and the event $\{\tau \geq t+1\} = \{\tau \leq t\}^c$ is in $\mathcal{F}_t$, we have

$$\begin{aligned}
\mathbb{E}[U_{(t+1)\wedge\tau} - U_{t\wedge\tau}|\mathcal{F}_t] &= \mathbb{E}[\mathbb{1}_{\{t+1\leq\tau\}}(U_{t+1} - U_t)|\mathcal{F}_t] \\
&= \mathbb{1}_{\{t+1\leq\tau\}}\mathbb{E}[U_{t+1} - U_t|\mathcal{F}_t] \\
&\leq 0
\end{aligned}$$

so the stopped process $(U_{t\wedge\tau})_{0\leq t\leq T}$ is also supermartingale.
(c) Now, the event

$$\{\tau_* > t\} = \{Y_0 < U_0, \ldots, Y_t < U_t\}$$

is $\mathcal{F}_t$-measurable since both Y and U are adapted, hence τ_* is a stopping time.
Since $U_t = \mathbb{E}(U_{t+1}|\mathcal{F}_t)$ on the event $\{t+1 \leq \tau_*\}$ we have

$$\begin{aligned}
U_{(t+1)\wedge\tau_*} - U_{t\wedge\tau_*} &= \mathbb{1}_{\{t+1\leq\tau_*\}}(U_{t+1} - U_t) \\
&= \mathbb{1}_{\{t+1\leq\tau_*\}}[U_{t+1} - \mathbb{E}(U_{t+1}|\mathcal{F}_t)].
\end{aligned}$$

In particular $\mathbb{E}[Y_{\tau_*}] = \mathbb{E}[U_{\tau_*}] = \mathbb{E}[U_{T\wedge\tau_*}] = U_0$.

Problem 10. Let $(X_k)_{k\in K}$ be a collection of real-valued random variables, where K is an arbitrary (possibly uncountable) index set. Our aim is to show there exists a random variable Y taking values in $\mathbb{R} \cup \{+\infty\}$ such that

- $Y \geq X_k$ almost surely for all $k \in K$, and
- if $Z \geq X_k$ almost surely for all $k \in K$ then $Z \geq Y$ almost surely.

This will show that the $Y = \text{ess sup}_k X_k$ exists.
(a) Show that there is no loss assuming that $|X_k(\omega)| \leq 1$ for all (k, ω). Hint: Consider $\tilde{X}_k = \tan^{-1}(X_k)$.

From now on, assume $|X_k(\omega)| \leq 1$ for all (k, ω). Let $\mathcal{C}$ be the collection of all countable subsets of K. Let

$$x = \sup_{A\in\mathcal{C}} \mathbb{E}[\sup_{k\in A} X_k]$$

Let $A_n \in \mathcal{C}$ be such that $\mathbb{E}[\sup_{k\in A_n} X_k] > x - 1/n$ and let $B = \cup_n A_n$. Let $Y = \sup_{k\in B} X_k$.
(b) Why is Y a random variable, i.e. measurable? Show that $\mathbb{E}(Y) = x$.
(c) Pick a $k \in K$, and let $Y_k = \max\{Y, X_k\} = \sup_{h\in B\cup\{k\}} X_h$. Show that $\mathbb{E}(Y_k) = x$. Why does this imply that $X_k \leq Y$ almost surely?
(d) Let Z be a random variable such that $Z \geq X_k$ a.s. for all $k \in K$. Prove that $Z \geq Y$ a.s.

Solution 10. (a) Suppose we know that every family of uniformly bounded random variables (indexed by some arbitrary index set) has an essential supremum. So given the family $(X_k)_{k\in K}$ let $\hat{X}_k = \tan^{-1}(X_k)$. Then the $\hat{X}_k$ are bounded so there exists $\hat{Y} = \text{ess sup}_k \hat{X}_k$.

Let $Y = \tan \hat{Y}$. Since the function tan is strictly increasing it is easy to see that Y has the properties characterising the essential supremum of $(X_k)_k$.
(b) Since A_n is countable, the function $Y_n = \sup_{k \in C_n} X_k$ is measurable for each n. Also $\cup_n A_n = B$ is countable and hence Y is also a random variable. For instance, note

$$\{Y > b\} = \cup_{k \in B} \{X_k > b\}.$$

Now, by replacing A_n with $\cup_{i=1}^n A_i$ we may assume $A_{n-1} \subseteq A_n$ for all $n \geq 1$, and hence we have $Y = \sup_n Y_n = \lim_n Y_n$. The result follows from the bounded convergence theorem.
(c) Since $Y_k \geq Y$ we have $\mathbb{E}(Y_k) \geq \mathbb{E}(Y) = x$. On the other hand,

$$\begin{aligned} x &= \sup_{A \in \mathcal{C}} \mathbb{E}[\sup_{h \in A} X_h] \\ &\geq \mathbb{E}[\sup_{h \in B \cup \{k\}} X_h] \\ &= \mathbb{E}(Y_k). \end{aligned}$$

Since $Y - Y_k \geq 0$ a.s. and $\mathbb{E}(Y - Y_k) = 0$, the pigeonhole principle implies $Y = Y_k$ a.s. Since $\max\{Y, X_k\} = Y$ we have $X_k \leq Y$.
(d) If $\mathbb{P}(Z \geq X_k) = 1$ for all $k \in K$, then

$$\mathbb{P}(Z \geq Y) = \mathbb{P}(\cap_{k \in B} \{Z \geq \hat{X}_k\}) = 1$$

since B is countable.