Advanced Financial Models Michael Tehranchi

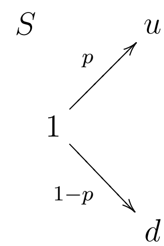
Example sheet 1 - Michaelmas 2010

Problem 1. In a one-period model, a numéraire asset is called risk-free if its time-1 price is not random: if asset i is risk-free, then $S_1^i = (1 + r^i)S_0^i$ for a real constant $r^i > -1$, called the risk-free rate of return. Suppose that a market model has at least one risk-free asset. Show that if there is no arbitrage, then the risk-free rate of return is unique, in the sense that if both asset i and asset j are risk-free then $r^i = r^j$.

Problem 2. (Binomial model) Consider a market with two assets. Asset 0 is a risk-free bond with

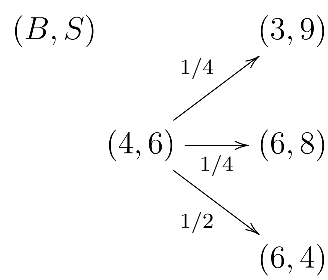
$$B_0 = 1, \text{ and } B_1 = 1 + r$$

for some constant interest rate $r > -1$, while asset 1 is a stock with prices



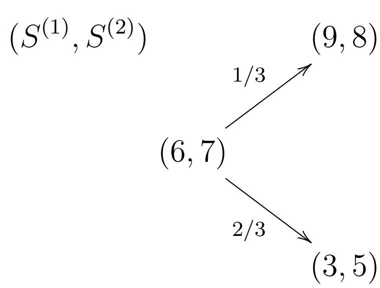
for some constant rates of return $d < u$ and a probability $0 < p < 1$. Use the definition of arbitrage to show that there is no arbitrage if and only if $d < 1 + r < u$. When there is no arbitrage, find all equivalent martingale measures $\mathbb{Q}$ (relative to asset 0, as usual) and their densities $\frac{d\mathbb{Q}}{d\mathbb{P}}$ with respect to the objective measure $\mathbb{P}$.

Problem 3. Consider a two-asset model with prices given by



Is there arbitrage in this market? If not, find all pricing kernels.

Problem 4. Consider a three-asset model with asset 0 cash $B_0 = B_1 = 1$ and assets 1 and 2 given by



Is there arbitrage in this market? If not, find all equivalent martingale measures.

Problem 5. * Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let Z be a random variable such that $Z > 0$ a.s. and $\mathbb{E}(Z) = 1$. Define a set function $\mathbb{Q}$ on $\mathcal{F}$ by

$$\mathbb{Q}(A) = \mathbb{E}(Z\mathbb{1}_A).$$

Show that $\mathbb{Q}$ is a probability measure which is equivalent to $\mathbb{P}$.

Problem 6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let W be a random vector with the d-dimensional normal $N_d(0, I)$ distribution, where I is the $d \times d$ identity matrix. Fix a constant vector $\alpha \in \mathbb{R}^d$ and define an equivalent measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\alpha \cdot W - |\alpha|^2/2}.$$

Prove that the random variable $\hat{W} = W - \alpha$ has the $N_d(0, I)$ distribution under $\mathbb{Q}$.

Problem 7. Consider a $d + 1$ asset model, where asset 0 is cash $B_0 = 1 = B_1$ while assets $1, \ldots, d$ have time-1 prices S_1 with the $N_d(\mu, V)$ distribution, where $\mu \in \mathbb{R}^d$ and V is a positive semi-definite $d \times d$ matrix. Use the previous problem to show that there is no arbitrage if V is non-singular. Show that there might be arbitrage (depending on the values of μ and S_0) if V is singular.

Problem 8. (Stiemke's theorem) Let A be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^T y \neq 0$.

What does this have to do with finance?

Problem 9. Let $X_1, \ldots, X_d$ be d random variables. We aim to show that if

$$a \cdot X \geq 0 \text{ a.s. for some } a \in \mathbb{R}^d \Rightarrow a \cdot X = 0 \text{ a.s.}$$

then there exists a random variable $Z > 0$ a.s. such that $\mathbb{E}(Z|X_i|) < \infty$ and $\mathbb{E}(ZX_i) = 0$ for all i.

(1) Why does the above theorem imply the hard direction of the 1FTAP?
(2) Why is there no loss of generality in assuming that the random variables $X_1, \ldots, X_d$ are linearly independent in the sense that if $b \cdot X = 0$ a.s. then $b = 0$.
(3) Let $F(a) = \mathbb{E}(e^{a \cdot X - |X|^2})$. Show that F is finite-valued and smooth.
(4) We aim now to show that there exists $a^* \in \mathbb{R}^d$ which minimizes F. Assuming this for the moment, show that $Z = e^{a^* \cdot X - |X|^2}$ satisfies the conclusions of the statement.
(5) Now suppose for the sake of finding a contradiction that F does not achieve its minimum. Let $(a_n)_n$ be such that $F(a_n) \downarrow \inf_a F(a)$. Why can we assume that $(a_n)_n$ is unbounded?
(6) Let $\hat{a}_n = a_n/|a_n|$. Why does the sequence $(\hat{a}_n)$ have a convergent subsequence?
(7) Suppose $|a_n| \to \infty$ and $\hat{a}_n \to \alpha$. Show that $\mathbb{P}(\alpha \cdot X > 0) = 0$.
(8) Use the hypothesis of the statement to show $\alpha \cdot X = 0$ a.s. Why is this our desired contradiction?

Problem 10. Let $X_1, \ldots, X_d$ and Y random variables, and let

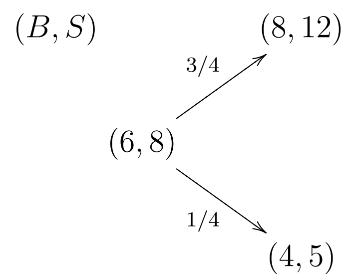
$$\mathcal{Z} = \{Z > 0 \text{ a.s.} : \mathbb{E}(Z|X_i|) < \infty \text{ and } \mathbb{E}(ZX_i) = 0 \text{ for all } i\} \neq \emptyset.$$

We aim to show that if $\mathbb{E}(ZY) = 0$ for all $Z \in \mathcal{Z}$ such that $\mathbb{E}(Z|Y|) < \infty$, then

$$Y = a \cdot X \text{ a.s. for some } a \in \mathbb{R}^d.$$

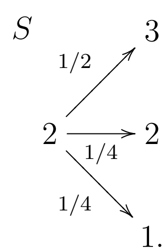
(1) Why does this statement imply the hard direction of the characterization of attainable claims?
(2) How can we modify the argument in the previous problem to find random variables $Z_r \in \mathcal{Z}$ of the form
$$Z_r = e^{a_r \cdot X + rY - |X|^2 - Y^2}$$
for $r \in \{0, 1\}$.
(3) Show that $Y = b \cdot X + \log(Z_1/Z_0)$ for some $b \in \mathbb{R}^d$.
(4) Use Jensen's inequality to show
$$\mathbb{E}\left[Z_0 \log \frac{Z_1}{Z_0}\right] \leq \mathbb{E}(Z_0) \log \frac{\mathbb{E}(Z_1)}{\mathbb{E}(Z_0)}$$
and deduce $\mathbb{E}(Z_1) \geq \mathbb{E}(Z_0)$.
(5) Find a similar argument to show $\mathbb{E}(Z_0) \geq \mathbb{E}(Z_1)$.
(6) Hence $\mathbb{E}(Z_1) = \mathbb{E}(Z_0)$. Use the strict concavity of the logarithm function to show that $Z_1 = Z_0$ a.s. Why does this imply the conclusion of the statement?

Problem 11. Consider a the binomial two-asset model with prices



Introduce a call option with strike 10, with payout $\xi_1 = (S_1 - 10)^+$. Find its unique no-arbitrage price ξ_0. What is the replicating portfolio for this option?

Problem 12. Consider a trinomial two-asset model with $B_0 = B_1 = 1$ and S given by



Find all equivalent martingale measures for this model. Now introduce a call option with strike 2, with payout $\xi_1 = (S_1 - 2)^+$. Show directly that the payout ξ_1 cannot be replicated by trading in the stock and bond. Prove that there is no arbitrage if and only if $0 < \xi_0 < \frac{1}{2}$.

Find the attainable random variable $X^* = x^* + \pi^* \cdot (S_1 - S_0)$ which minimizes expected square hedging error $\mathbb{E}[(X - \xi_1)^2]$. Find the random variable ρ^* that minimizes $\rho \mapsto \mathbb{E}[\rho^2]$ subject to $\mathbb{E}[\rho(B_1, S_1)] = (B_0, S_0)$, and verify $x^* = \mathbb{E}(Z^* \xi_1)$.

Problem 13. Let $\mathcal{Z}$ be the set of Radon–Nikodym derivatives of equivalent martingale measures for a given market model. Prove that $\mathcal{Z}$ is a convex subset of L^1. Show by example that $\mathcal{Z}$ is *not* necessarily closed in L^1.

(Remember that a random variable Z is in L^1 iff $\mathbb{E}(|Z|) < \infty$, and that a subset $\mathcal{Z}$ of L^1 is closed in L^1 iff for every sequence $Z_1, Z_2, \ldots$ of random variables in $\mathcal{Z}$ such that $\mathbb{E}(|Z_n - Z|) \to 0$ for some random variable $Z \in L^1$, the limit point Z is also in $\mathcal{Z}$.)

Problem 14. Now consider a two-asset model (B, S), where asset 0 is a risk-free bond with risk-free (not random) rate of return r, given by

$$r = \frac{B_1}{B_0} - 1,$$

and where asset 1 is a stock with expected rate of return R given by

$$R = \frac{\mathbb{E}(S_1)}{S_0} - 1.$$

(Assume S_1 is integrable.) Given his initial wealth X_0, an investor maximizes the expected utility $\mathbb{E}[U(X_1)]$ for a given increasing, concave utility function U. Assume that the maximum is attained when he holds π^* shares of the stock. Show that if $\pi^* > 0$, then $R \geq r$. Conversely, show that if $R > r$ then $\pi^* \geq 0$. Do these results agree with your intuition?

(This is harder: Suppose that U is strictly concave and that S_1 is not constant. Show that if $\pi^* > 0$ then $R > r$. Can you find an example where $R > r$, but $\pi^* = 0$?)