Advanced Financial Models Michael Tehranchi
Example sheet 1 - Michaelmas 2019

Problem 1. Consider a market with $n = 1$ asset with prices $(P_t)_{t\geq 0}$ and dividends $(\delta_t)_{t\geq 1}$. Suppose there is no arbitrage.
(a) Show that there exists a strictly positive adapted process Y such that

$$P_t = \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{T} Y_u \delta_u | \mathcal{F}_t\right) + \frac{1}{Y_t}\mathbb{E}(Y_T P_T | \mathcal{F}_t)$$

for all $0 \leq t \leq T$.
(b) Now suppose that $P_t \geq 0$ and $\delta_t \geq 0$ almost surely for all t. Letting Y be the process in part (a), show that

$$P_t \geq \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{\infty} Y_u \delta_u | \mathcal{F}_t\right)$$

for all $t \geq 0$. Find a condition on P, Y and δ such that there is equality in the above inequality. [The right-hand side of the inequality could be thought of the fundamental value of the asset - i.e. the present discounted value of the stream of dividend payments. When there is strict inequality, the price of the asset is strictly greater than its fundamental value, modelling a price bubble. Note that no arbitrage does not forbid such price bubbles.]

Problem 2. Consider an arbitrage-free market with $n = 1$ asset with prices $(P_t)_{t\geq 0}$ and dividends $(\delta_t)_{t\geq 1}$. Suppose that $\delta_t \geq 0$ almost surely for all $t \geq 1$, and that there exists a non-random time $T > 0$ such that $P_T \geq 0$ almost surely and $\mathbb{P}(P_T > 0) > 0$. Show that $P_0 > 0$. Must it be the case that $P_t > 0$ almost surely for all $0 \leq t < T$?

Problem 3. Given a filtration $(\mathcal{F}_t)_{t\geq 0}$ let $(Z_t)_{t\geq 1}$ be a non-negative adapted integrable process such that $\mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1$ for all $t \geq 1$. Let

$$M_t = \prod_{s=1}^{t} Z_s.$$

Show that M is a martingale. [Hint: First show that M is a local martingale.]

Problem 4. Consider a discrete time model with a single asset with positive prices $(P_t)_{t\geq 0}$ and non-negative dividends $(\delta_t)_{t\geq 1}$. Show that there is a self-financing (pure-investment) trading strategy with corresponding wealth process

$$Q_t = P_t \prod_{s=1}^{t}\left(1 + \frac{\delta_s}{P_s}\right).$$

Let Y be a positive adapted process. Show that

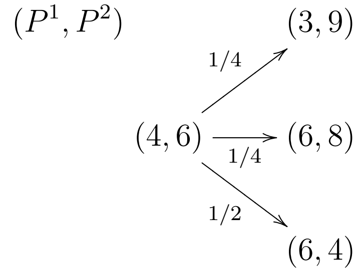
$$M_t = Y_t P_t + \sum_{s=1}^{t} Y_s \delta_s$$

defines a martingale if and only if

$$N_t = Y_t Q_t$$

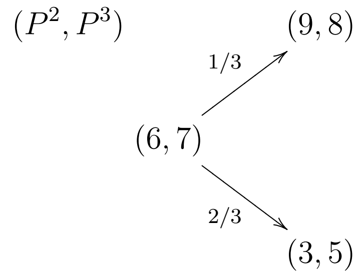
defines a martingale.

Problem 5. Consider a two-asset model with no dividends and prices given by



Is there arbitrage in this market? If so, find all arbitrages. If not, find all pricing kernels.

Problem 6. Consider a three-asset model with no dividends such that asset 1 is cash (meaning $P_0^1 = P_1^1 = 1$) and assets 2 and 3 given by



Is there arbitrage in this market? If so, find all arbitrages. If not, find all pricing kernels.

Problem 7. * Let X be a martingale, K a predictable process, and M_0 a constant. Let

$$M_t = M_0 + \sum_{s=1}^{t} K_s(X_s - X_{s-1}).$$

Show that if M_T is integrable for some non-random time $T > 0$, then $(M_t)_{0 \le t \le T}$ is a true martingale. Hint: Show that M_{T-1} is integrable.

Problem 8. This problem leads you through an alternative proof of the one-period 1FTAP using the following version of the separating hyperplane theorem: Let $C \subset \mathbb{R}^n$ be convex and $x \in \mathbb{R}^n$ not contained in C. Then there exists a $\lambda \in \mathbb{R}^n$ such that $\lambda \cdot (y - x) \ge 0$ for all $y \in C$, where the inequality is strict for at least one $y \in C$.

We are given a market model $(P_t)_{t \in \{0,1\}}$. (Without loss, assume $\delta_1 = 0$.)

(a) Define a collection of random variables by

$$\mathcal{Z} = \{Z : Z > 0 \text{ a.s. and } \mathbb{E}(Z\|P_1\|) < \infty.$$

Show that $\mathcal{Z}$ not empty and convex.

(b) Now define a subset of $\mathbb{R}^n$ by

$$\mathcal{P} = \{\mathbb{E}(ZP_1) : Z \in \mathcal{Z}\}.$$

Show that $\mathcal{P}$ is not empty and convex. Furthermore, show that if $P_0 \in \mathcal{P}$ there exists a pricing kernel.

For the rest of the problem assume $P_0 \notin \mathcal{P}$. We must find an arbitrage.

(c) Use the given separating hyperplane theorem to show that there exists a vector $H \in \mathbb{R}^n$ such that $\mathbb{E}(ZH \cdot P_1) \ge H \cdot P_0$ for all $Z \in \mathcal{Z}$ with strict inequality for all at least one $Z \in \mathcal{Z}$.

(d) Use the conclusion of part (c) to show $H \cdot P_0 \le 0$. [Hint: fix an element $Z_0 \in \mathcal{Z}$, and let $Z = \varepsilon Z_0$. Now look at the inequality when $\varepsilon \downarrow 0$.]
(e) Let $A = \{H \cdot P_1 < 0\}$. By setting $Z = (\frac{1}{\varepsilon}\mathbb{1}_A + 1)Z_0$, show $\mathbb{P}(A) = 0$.
(f) Finally, by appealing to the conclusion of part (c), show that H is an arbitrage.

Problem 9. (Stiemke's theorem) Let A be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \dots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^\top y)_i \ge 0$ for all $i = 1, \dots, n$ such that $A^\top y \ne 0$.

What does this have to do with finance?

Problem 10. * Consider an arbitrage-free market with at least two assets, where no asset pays a dividend.
(a) Prove the *law of one price*: if $P_T^1 = P_T^2$ almost surely, where $T > 0$ is a non-random time, then $P_t^1 = P_t^2$ almost surely for all $0 \le t \le T$.
(b) Find an example of an arbitrage-free market for which there is a stopping time τ such that $P_\tau^1 = P_\tau^2$ almost surely, and yet $P_0^1 \ne P_0^2$.

Problem 11. (Tower property of conditional expectation) Let X and Y be identically distributed random variables taking values in the set $\{2^n : n \ge 0\}$ such that $X/Y \in \{1/2, 2\}$ almost surely and

$$\mathbb{P}(X = 2^n, Y = 2^{n+1}) = \frac{1}{4}2^{-n} = \mathbb{P}(X = 2^{n+1}, Y = 2^n) \quad \text{for } n \ge 0.$$

(a) Show that $\mathbb{P}(X = 1) = \frac{1}{4}$ and

$$\mathbb{P}(X = 2^n) = \frac{3}{4}2^{-n} \quad \text{for } n \ge 1.$$

(b) Show that $\mathbb{P}(Y = 2|X = 1) = 1$ and

$$\mathbb{P}(Y = 2^{n+1}|X = 2^n) = \frac{1}{3} = 1 - \mathbb{P}(Y = 2^{n-1}|X = 2^n) \quad \text{for } n \ge 1.$$

(c) Let $Z = Y - X$. Show that $\mathbb{E}(Z|X = 1) = 1$ and

$$\mathbb{E}(Z|X = 2^n) = 0 \quad \text{for } n \ge 1.$$

(d) From part (c) we have $\mathbb{E}(Z|X) = \mathbb{1}_{\{X=1\}}$ and hence

$$\mathbb{E}(Z) = \mathbb{E}[\mathbb{E}(Z|X)] = \frac{1}{4} > 0.$$

However, by symmetry we also have $\mathbb{E}(Z|Y) = -\mathbb{1}_{\{Y=1\}}$ and

$$\mathbb{E}(Z) = \mathbb{E}[\mathbb{E}(Z|Y)] = -\frac{1}{4} < 0.$$

What has gone wrong?!