Problem 1. Consider a market with \( n = 1 \) asset with prices \((P_t)_{t \geq 0}\) and dividends \((\delta_t)_{t \geq 1}\). Suppose there is no arbitrage.

(a) Show that there exists a strictly positive adapted process \( Y \) such that
\[
P_t = \frac{1}{Y_t} \mathbb{E}_t \left( \sum_{u=t+1}^{T} Y_u \delta_u | \mathcal{F}_t \right) + \frac{1}{Y_t} \mathbb{E}_t (Y_T P_T | \mathcal{F}_t)
\]
for all \( 0 \leq t \leq T \).

(b) Now suppose that \( P_t \geq 0 \) and \( \delta_t \geq 0 \) almost surely for all \( t \). Letting \( Y \) be the process in part (a), show that
\[
P_t \geq \frac{1}{Y_t} \mathbb{E}_t \left( \sum_{u=t+1}^{\infty} Y_u \delta_u | \mathcal{F}_t \right)
\]
for all \( t \geq 0 \). Find a condition on \( P, Y \) and \( \delta \) such that there is equality in the above inequality. [The right-hand side of the inequality could be thought of the fundamental value of the asset - i.e. the present discounted value of the stream of dividend payments. When there is strict inequality, the price of the asset is strictly greater than its fundamental value, modelling a price bubble. Note that no arbitrage does not forbid such price bubbles.]

Problem 2. Consider an arbitrage-free market with \( n = 1 \) asset with prices \((P_t)_{t \geq 0}\) and dividends \((\delta_t)_{t \geq 1}\). Suppose that \( \delta_t \geq 0 \) almost surely for all \( t \geq 1 \), and that there exists a non-random time \( T > 0 \) such that \( P_T \geq 0 \) almost surely and \( \mathbb{P}(P_T > 0) > 0 \). Show that \( P_0 > 0 \). Must it be the case that \( P_t > 0 \) almost surely for all \( 0 \leq t < T \)?

Problem 3. Given a filtration \((\mathcal{F}_t)_{t \geq 0}\) let \((Z_t)_{t \geq 1}\) be a non-negative adapted integrable process such that \( \mathbb{E}(Z_t | \mathcal{F}_{t-1}) = 1 \) for all \( t \geq 1 \). Let
\[
M_t = \prod_{s=1}^{t} Z_s.
\]
Show that \( M \) is a martingale. [Hint: First show that \( M \) is a local martingale.]

Problem 4. Consider a discrete time model with a single asset with positive prices \((P_t)_{t \geq 0}\) and non-negative dividends \((\delta_t)_{t \geq 1}\). Show that there is a self-financing (pure-investment) trading strategy with corresponding wealth process
\[
Q_t = P_t \prod_{s=1}^{t} \left( 1 + \frac{\delta_s}{P_s} \right).
\]

Let \( Y \) be a positive adapted process. Show that
\[
M_t = Y_t P_t + \sum_{s=1}^{t} Y_s \delta_s
\]
defines a martingale if and only if
\[
N_t = Y_t Q_t
\]
defines a martingale.
Problem 5. Consider a two-asset model with no dividends and prices given by
\[
(P^1, P^2) \quad (3, 9)
\]
\[
\begin{align*}
(4, 6) & \quad \rightarrow \quad (6, 8) \\
\quad \frac{1}{4} & \\
(6, 4) & \quad \rightarrow \quad (1, 2) \\
\quad \frac{1}{2}
\end{align*}
\]
Is there arbitrage in this market? If so, find all arbitrages. If not, find all pricing kernels.

Problem 6. Consider a three-asset model with no dividends such that asset 1 is cash (meaning \( P^1_0 = P^1_1 = 1 \)) and assets 2 and 3 given by
\[
(P^2, P^3) \quad (9, 8)
\]
\[
\begin{align*}
(6, 7) & \quad \rightarrow \quad (3, 5) \\
\quad \frac{2}{3}
\end{align*}
\]
Is there arbitrage in this market? If so, find all arbitrages. If not, find all pricing kernels.

Problem 7. * Let \( X \) be a martingale, \( K \) a predictable process, and \( M_0 \) a constant. Let
\[
M_t = M_0 + \sum_{s=1}^{t} K_s (X_s - X_{s-1}).
\]
Show that if \( M_T \) is integrable for some non-random time \( T > 0 \), then \( (M_t)_{0 \leq t \leq T} \) is a true martingale. Hint: Show that \( M_{T-1} \) is integrable.

Problem 8. This problem leads you through an alternative proof of the one-period 1FTAP using the following version of the separating hyperplane theorem: Let \( C \subset \mathbb{R}^n \) be convex and \( x \in \mathbb{R}^n \) not contained in \( C \). Then there exists a \( \lambda \in \mathbb{R}^n \) such that \( \lambda \cdot (y - x) \geq 0 \) for all \( y \in C \), where the inequality is strict for at least one \( y \in C \).

We are given a market model \( (P_t)_{t \in \{0, 1\}} \). (Without loss, assume \( \delta_1 = 0 \).)
(a) Define a collection of random variables by
\[
Z = \{ Z : Z > 0 \text{ a.s. and } \mathbb{E}(Z \| P_1 \|) < \infty \}.
\]
Show that \( Z \) not empty and convex.
(b) Now define a subset of \( \mathbb{R}^n \) by
\[
\mathcal{P} = \{ \mathbb{E}(Z P_1) : Z \in Z \}.
\]
Show that \( \mathcal{P} \) is not empty and convex. Furthermore, show that if \( P_0 \in \mathcal{P} \) there exists a pricing kernel.

For the rest of the problem assume \( P_0 \notin \mathcal{P} \). We must find an arbitrage.
(c) Use the given separating hyperplane theorem to show that there exists a vector \( H \in \mathbb{R}^n \) such that \( \mathbb{E}(ZH \cdot P_1) \geq H \cdot P_0 \) for all \( Z \in Z \) with strict inequality for all at least one \( Z \in Z \).
(d) Use the conclusion of part (c) to show $H \cdot P_0 \leq 0$. [Hint: fix an element $Z_0 \in \mathcal{Z}$, and let $Z = \varepsilon Z_0$. Now look at the inequality when $\varepsilon \downarrow 0$.]
(e) Let $A = \{H \cdot P_1 < 0\}$. By setting $Z = (\frac{1}{\varepsilon} 1_A + 1)Z_0$, show $\mathbb{P}(A) = 0$. 
(f) Finally, by appealing to the conclusion of part (c), show that $H$ is an arbitrage.

**Problem 9.** (Stiemke’s theorem) Let $A$ be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^\top y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^\top y \neq 0$.

What does this have to do with finance?

**Problem 10.** * Consider an arbitrage-free market with at least two assets, where no asset pays a dividend.
(a) Prove the law of one price: if $P_T^1 = P_T^2$ almost surely, where $T > 0$ is a non-random time, then $P_t^1 = P_t^2$ almost surely for all $0 \leq t \leq T$.
(b) Find an example of an arbitrage-free market for which there is a stopping time $\tau$ such that $P_\tau^1 = P_\tau^2$ almost surely, and yet $P_0^1 \neq P_0^2$.

**Problem 11.** (Tower property of conditional expectation) Let $X$ and $Y$ be identically distributed random variables taking values in the set $\{2^n : n \geq 0\}$ such that $X/Y \in \{1/2, 2\}$ almost surely and

$$
\mathbb{P}(X = 2^n, Y = 2^{n+1}) = \frac{1}{4}2^{-n} = \mathbb{P}(X = 2^{n+1}, Y = 2^n) \quad \text{for } n \geq 0.
$$

(a) Show that $\mathbb{P}(X = 1) = \frac{1}{4}$ and

$$
\mathbb{P}(X = 2^n) = \frac{3}{4}2^{-n} \quad \text{for } n \geq 1.
$$

(b) Show that $\mathbb{P}(Y = 2|X = 1) = 1$ and

$$
\mathbb{P}(Y = 2^{n+1}|X = 2^n) = \frac{1}{3} = 1 - \mathbb{P}(Y = 2^{n-1}|X = 2^n) \quad \text{for } n \geq 1.
$$

(c) Let $Z = Y - X$. Show that $\mathbb{E}(Z|X = 1) = 1$ and

$$
\mathbb{E}(Z|X = 2^n) = 0 \quad \text{for } n \geq 1.
$$

(d) From part (c) we have $\mathbb{E}(Z|X) = \mathbbm{1}_{\{X = 1\}}$ and hence

$$
\mathbb{E}(Z) = \mathbb{E}[\mathbb{E}(Z|X)] = \frac{1}{4} > 0.
$$

However, by symmetry we also have $\mathbb{E}(Z|Y) = -\mathbbm{1}_{\{Y = 1\}}$ and

$$
\mathbb{E}(Z) = \mathbb{E}[\mathbb{E}(Z|Y)] = -\frac{1}{4} < 0.
$$

What has gone wrong?!