Advanced Financial Models Michael Tehranchi

Example sheet 1 - Michaelmas 2010

Problem 1. In a one-period model, a numéraire asset is called risk-free if its time-1 price is not random: if asset i is risk-free, then $S_1^i = (1 + r^i)S_0^i$ for a real constant $r^i > -1$, called the risk-free rate of return. Suppose that a market model has at least one risk-free asset. Show that if there is no arbitrage, then the risk-free rate of return is unique, in the sense that if both asset i and asset j are risk-free then $r^i = r^j$.

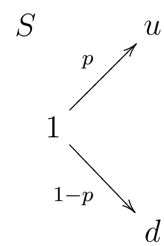
Solution 1. We will show that there is an arbitrage if $r^i < r^j$. Indeed, simply let $\pi^i = -1/S_0^i$ and $\pi^j = 1/S_0^j$ so that

$$X_0 = \pi^i S_0^i + \pi^j S_0^j = 0 \text{ and } X_1 = \pi^i S_1^i + \pi^j S_1^j = r^j - r^i > 0$$

Problem 2. (Binomial model) Consider a market with two assets. Asset 0 is a risk-free bond with

$$B_0 = 1, \text{ and } B_1 = 1 + r$$

for some constant interest rate $r > -1$, while asset 1 is a stock with prices



for some constant rates of return $d < u$ and a probability $0 < p < 1$. Use the definition of arbitrage to show that there is no arbitrage if and only if $d < 1 + r < u$. When there is no arbitrage, find all equivalent martingale measures $\mathbb{Q}$ (relative to asset 0, as usual) and their densities $\frac{d\mathbb{Q}}{d\mathbb{P}}$ with respect to the objective measure $\mathbb{P}$.

Solution 2. For a portfolio (ϕ, π), the wealth process satisfies $X_0 \leq 0$ and $X_1 \geq 0$ a.s. if and only if the following inequalities hold

$$\begin{array}{lrcl} (1) & \phi + \pi & \leq & 0 \\ (2) & \phi(1+r) + \pi u & \geq & 0 \\ (3) & \phi(1+r) + \pi d & \geq & 0. \end{array}$$

Suppose $1+r \geq u > d$. Then the portfolio $\phi = 1$, $\pi = -1$ is an arbritrage since inequalities (1), (2), and (3) all hold, and inequality (2) is strict.

Similarly, if $1 + r \leq d < u$, then the portfolio $\phi = -1$, $\pi = 1$ is an arbitrage.

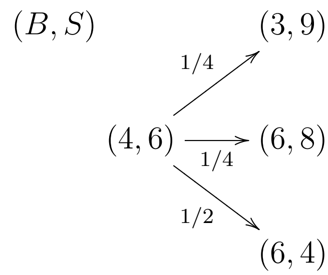
Finally, suppose $d < 1+r < u$. Subtracting $1+r > 0$ times inequality (1) from inequality (2) yields $\pi[u - (1 + r)] \geq 0$, and dividing by $u - (1 + r) > 0$ yields $\pi \geq 0$. Similarly, the inequality (2)-$(1+r)\times$ (1) yields $\pi \leq 0$, so $\pi = 0$. Plugging this into (1) and (2) then yields $\phi = 0$, so there is no arbitrage.

In the case where no arbitrage is possible ($d < 1+r < u$) we know by the first FTAP that there exists at least one equivalent martingale measure $\mathbb{Q}$, under which the discounted stock price $\tilde{S} = S/B$ is a martingale. If we call the two outcomes H and T, such that $S_1(H) = u$ and $S_1(T) = d$, and let $q = \mathbb{Q}\{H\}$, we must have $\mathbb{E}(\tilde{S}_1) = q\frac{u}{1+r} + (1-q)\frac{d}{1+r} = \tilde{S}_0 = 1$, i.e.

$q = \frac{1+r-d}{u-d}$. Note that the no-arbitrage condition gives that $q \in (0,1)$. The density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is simply the likelihood ratios of both measures:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(H) = \frac{1+r-d}{p(u-d)} \text{ and } \frac{d\mathbb{Q}}{d\mathbb{P}}(T) = \frac{u-r-1}{(1-p)(u-d)}.$$

Problem 3. Consider a two-asset model with prices given by



Is there arbitrage in this market? If not, find all pricing kernels.

Solution 3. Notice that

$$\tilde{S}_1 - \tilde{S}_0 = \begin{cases} 3/2 & \text{with prob. } 1/4 \\ -1/6 & \text{with prob. } 1/4 \\ -5/6 & \text{with prob. } 1/2 \end{cases}$$

and hence if $\pi(\tilde{S}_1 - \tilde{S}_0) \geq 0$ almost surely, then $\pi = 0$. This means there is no arbitrage.

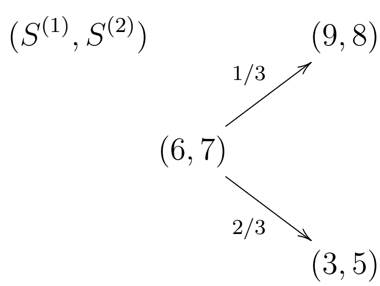
A pricing kernel can be found by solving $\mathbb{E}[\rho(B_1, S_1)] = (B_0, S_0)$ for ρ. Letting the values ρ takes be $a, b, c > 0$ for up/middle/down outcomes respectively, we have

$$\begin{array}{ccccccc} \frac{1}{4}3a & + & \frac{1}{4}6b & + & \frac{1}{2}6c & = & 4 \\ \frac{1}{4}9a & + & \frac{1}{4}8b & + & \frac{1}{2}4c & = & 6 \end{array}$$

whose solution can be written as

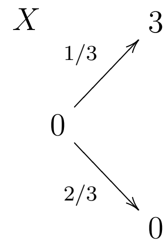
$$(a,b,c) = \left(\frac{8}{15} + \frac{48}{35}u, \frac{12}{5}(1-u), \frac{6}{7}u\right) \text{ for } 0 < u < 1.$$

Problem 4. Consider a three-asset model with asset 0 cash $B_0 = B_1 = 1$ and assets 1 and 2 given by



Is there arbitrage in this market? If not, find all equivalent martingale measures.

Solution 4. Yes, the portfolio $\bar{\pi} = (9, 2, -3)$ is an arbitrage since the corresponding wealth process



has $X_0 = 0$, $X_1 \geq 0$ a.s., and $\mathbb{P}(X_1 > 0) = 1/3 > 0$

Problem 5. * Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let Z be a random variable such that $Z > 0$ a.s. and $\mathbb{E}(Z) = 1$. Define a set function $\mathbb{Q}$ on $\mathcal{F}$ by

$$\mathbb{Q}(A) = \mathbb{E}(Z \mathbb{1}_A).$$

Show that $\mathbb{Q}$ is a probability measure which is equivalent to $\mathbb{P}$.

Solution 5. This is a technical exercise to give you some practice with measure theory. First we show that $\mathbb{Q}$ is a countably additive set function. Since $\mathbb{1}_\emptyset(\omega) = 0$ for all $\omega \in \Omega$, we have

$$\mathbb{Q}(\emptyset) = \mathbb{E}(0) = 0.$$

Now let $A_1, A_2, \ldots$ be a collection of disjoint events in $\mathcal{F}$. Note that

$$\sum_{i=1}^{N} \mathbb{1}_{A_i}(\omega) = \mathbb{1}_{\bigcup_{i=1}^{N} A_i}(\omega)$$

for all $\omega \in \Omega$ since the A_i's are disjoint, with the equality still valid if $N = \infty$. Hence

$$\begin{aligned} \mathbb{Q}(\bigcup_{i=1}^{\infty} A_i) &= \mathbb{E}(Z \mathbb{1}_{\bigcup_{i=1}^{\infty} A_i}) \\ &= \mathbb{E}(\sum_{i=1}^{\infty} Z \mathbb{1}_{A_i}) \\ &= \sum_{i=1}^{\infty} \mathbb{Q}(A_i) \end{aligned}$$

We have now established that $\mathbb{Q}$ is a measure, though we should justify the interchange of expectation and summation above. That is, if $X_i \geq 0$ a.s. then we should explain why

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i).$$

There are several ways of doing this. One way is to let $Y_N = \sum_{i=1}^{N} X_i$. Then the claim is just the monotone convergence theorem

$$\mathbb{E}(\lim_N Y_N) = \lim_N \mathbb{E}(Y_N).$$

Another way to justify the interchange is via Tonelli's theorem applied to the product measure $\mathbb{P} \times \mu$ where μ is the counting measure on $\{1, 2, \ldots\}$.

Now since $\mathbb{1}_\Omega(\omega) = 1$ for all $\omega \in \Omega$ we have

$$\mathbb{Q}(\Omega) = \mathbb{E}(Z) = 1.$$

Hence, $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$.

It remains to show that $\mathbb{Q}$ is equivalent to $\mathbb{P}$. Suppose that $\mathbb{P}(A) = 1$. Then $\mathbb{1}_A(\omega) = 1$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Hence $\mathbb{Q}(A) = \mathbb{E}(Z\mathbb{1}_A) = 1$, so $\mathbb{P}(A) = 1 \Rightarrow \mathbb{Q}(A) = 1$.

Now suppose $\mathbb{Q}(A) = 1$. Then $\mathbb{E}(Z\mathbb{1}_A) = 1$. Since $\mathbb{E}(Z) = 1$, we have $\mathbb{E}(Z\mathbb{1}_{A^c}) = 0$. Since $Z\mathbb{1}_{A^c} \geq 0$ $\mathbb{P}$-a.s but has mean zero, we know that $Z\mathbb{1}_{A^c} = 0$ $\mathbb{P}$-a.s. But since $Z > 0$ $\mathbb{P}$-a.s., we must conclude $\mathbb{P}(A^c) = 0$. That is, we have just shown $\mathbb{Q}(A) = 1 \Rightarrow \mathbb{P}(A) = 1$.

Incidentally, we have used repeatedly the following basic fact: if $Y \geq 0$ a.s. and $\mathbb{E}(Y) = 0$, then $Y = 0$ a.s. How is this proven? Suppose $Y \geq 0$ a.s. and $\mathbb{P}(Y > 0) > 0$. Then we must prove $\mathbb{E}(Y) > 0$. Since

$$\mathbb{P}(Y > 0) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} \{Y \geq 1/k\}\right) = \lim_{k\to\infty} \mathbb{P}(Y \geq 1/k) > 0$$

there exists an $\epsilon > 0$ such that $\mathbb{P}(Y \geq \epsilon) > 0$. Hence

$$\mathbb{E}(Y) \geq \mathbb{E}(Y\mathbb{1}_{\{Y\geq\epsilon\}}) \geq \epsilon\mathbb{P}(Y \geq \epsilon) > 0$$

as desired.

Problem 6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let W be a random vector with the d-dimensional normal $N_d(0, I)$ distribution, where I is the $d \times d$ identity matrix. Fix a constant vector $\alpha \in \mathbb{R}^d$ and define an equivalent measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\alpha \cdot W - |\alpha|^2/2}.$$

Prove that the random variable $\hat{W} = W - \alpha$ has the $N_d(0, I)$ distribution under $\mathbb{Q}$.

Solution 6. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded and (Borel) measurable function. The following computation proves the claim

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[f(\hat{W})] &= \mathbb{E}^{\mathbb{P}}[e^{\alpha\cdot W - |\alpha|^2/2} f(W - \alpha)] \\
&= \int_{\mathbb{R}^d} e^{\alpha\cdot w - |\alpha|^2/2} f(w - \alpha) \frac{e^{-|w|^2/2}}{(2\pi)^{d/2}} dw \\
&= \int_{\mathbb{R}^d} f(w - \alpha) \frac{e^{-|w-\alpha|^2/2}}{(2\pi)^{d/2}} dw \\
&= \int_{\mathbb{R}^d} f(u) \frac{e^{-|u|^2/2}}{(2\pi)^{d/2}} du
\end{aligned}$$

where the last line follows from the change of variables $u = w - \alpha$.

Problem 7. Consider a $d+1$ asset model, where asset 0 is cash $B_0 = 1 = B_1$ while assets $1, \ldots, d$ have time-1 prices S_1 with the $N_d(\mu, V)$ distribution, where $\mu \in \mathbb{R}^d$ and V is a positive semi-definite $d \times d$ matrix. Use the previous problem to show that there is no arbitrage if V is non-singular. Show that there might be arbitrage (depending on the values of μ and S_0) if V is singular.

Solution 7. By assumption, the risky assets at time 1 can be written as a d-dimensional random vector the form $S_1 = V^{1/2}W + \mu$, where W has d-dimensional normal $N_d(0, I)$ distribution, i.e. W is an d-dimensional random vector whose components are all independent standard normal $N(0, 1)$,

Suppose first that V is non-singular (i.e. invertible). To prove that there is no arbitrage, it is enough to find an equivalent martingale measure $\mathbb{Q}$, i.e. a measure such that $\mathbb{E}^{\mathbb{Q}}(S_1) = S_0$. Writing $S_1 = V^{1/2}\hat{W} + S_0$ where

$$\hat{W} = W + V^{-1/2}(\mu - S_0),$$

we need only find a measure $\mathbb{Q}$ for which $\mathbb{E}^{\mathbb{Q}}(\hat{W}) = 0$. By Problem 1, we may choose

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{V^{-1/2}(S_0-\mu)\cdot W - |V^{-1/2}(S_0-\mu)|^2/2},$$

since $\hat{W}$ has the $N_d(0, I)$ under $\mathbb{Q}$.

Let's now look at the case where V is singular. In this case, there exists some nonzero vector α, such that $V\alpha = 0$. This means that $\alpha \cdot V\alpha = |V^{1/2}\alpha|^2 = 0$ and thus $V^{1/2}\alpha = 0$. Now, this gives us:

$$\alpha \cdot (S_1 - S_0) = \alpha \cdot V^{1/2}W + \alpha \cdot (\mu - S_0) = \alpha \cdot (\mu - S_0).$$

Therefore, we have found a trading strategy (namely α), such that $\alpha \cdot (S_1 - S_0)$ is a constant. So, if $\alpha \cdot (S_1 - S_0) > 0$ then α is an arbitrage; else, if $\alpha \cdot (S_1 - S_0) < 0$, then the portfolio $-\alpha$ is an arbitrage.

Note that if the vector $\mu - S_0$ is orthogonal to α, then the portfolio α is not an arbitrage.

Problem 8. (Stiemke's theorem) Let A be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^T y \neq 0$.

What does this have to do with finance?

Solution 8. Let $\Omega = \{1, \ldots, n\}$, $\mathcal{F}$ the set of all subsets of Ω, and $\mathbb{P}(\{j\}) = 1/n$ for all $j \in \Omega$. Let $B_0 = 1 = B_1$, and define the random variable $\Delta S : \Omega \to \mathbb{R}^m$ by the $(\Delta S)(j) = a_{\cdot,j}$ where $A = (a_{i,j})_{i,j}$.

An arbitrage is a portfolio $\pi \in \mathbb{R}^m$ such that $\pi \cdot \Delta S \geq 0$ almost surely, and $\mathbb{P}(\pi \cdot \Delta S > 0) > 0$. (Note that we do not need the $\sim$-notation since $B = 1$.) In this context an arbitrage can be identified with a vector $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^T y \neq 0$.

The density of a equivalent martingale measure is a strictly positive real-valued random variable Z with $\mathbb{E}(Z) = 1$ such that $\mathbb{E}(Z\Delta S) = 0$. In this context a density can be identified with an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n x_i = 1$ and such that $Ax = 0$.

The first fundamental theorem of asset pricing is that there exists an equivalent martingale measure if and only if no arbitrage exists.

We have seen one proof of the 1FTAP. For the sake of developing intuition, here's another one: As before, the easy direction is to show that the existence of an equivalent martingale measure implies no arbitrage: Suppose that there exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all

$i = 1, \dots, n$ such that $Ax = 0$. If there exists a $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \dots, n$ then $x \cdot (A^T y) = (Ax) \cdot y = 0$. Hence $A^T y = 0$.

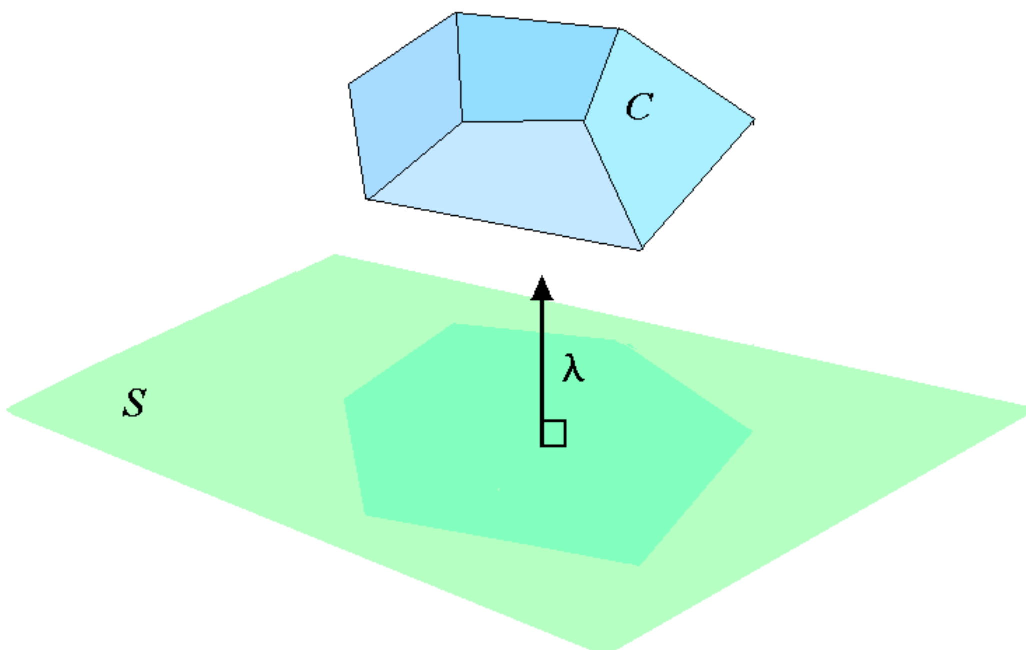
Now we prove the harder direction using a version of the separating hyperplane theorem. Suppose that there is no arbitrage: If there exists a $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \dots, n$ then $A^T y = 0$. Let

$$S = \{A^T y : y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

and let

$$C = \left\{ u \in \mathbb{R}^n : u_i \geq 0 \text{ for all } i = 1, \dots, n \text{ and } \sum_{i=1}^{n} u_i = 1 \right\} \subset \mathbb{R}^n.$$

By assumption, the subspace S and the convex compact set C are disjoint. The situation is illustrated by the figure. A version of the separating hyperplane theorem, says there exists



a vector $\lambda \in \mathbb{R}^n$ such that

$$\begin{aligned} \lambda \cdot v = 0 & \quad \text{for all } v \in S \\ \lambda \cdot u > & \quad \text{for all } u \in C. \end{aligned}$$

(Try proving this!) We will be done once we show that $\lambda_j > 0$ for all j. So fix a $j \in \{1, \dots, n\}$ and let $e \in \mathbb{R}^n$ be given by $e_j = 1$ and $e_i = 0$ for all $i \neq j$. Then $e \in C$ and $\lambda \cdot e = \lambda_j > 0$ as desired.

Problem 9. Let $X_1, \dots, X_d$ be d random variables. We aim to show that if

$$a \cdot X \geq 0 \text{ a.s. for some } a \in \mathbb{R}^d \Rightarrow a \cdot X = 0 \text{ a.s.}$$

then there exists a random variable $Z > 0$ a.s. such that $\mathbb{E}(Z|X_i|) < \infty$ and $\mathbb{E}(ZX_i) = 0$ for all i.

(1) Why does the above theorem imply the hard direction of the 1FTAP?

(2) Why is there no loss of generality in assuming that the random variables $X_1, \dots, X_d$ are linearly independent in the sense that if $b \cdot X = 0$ a.s. then $b = 0$.
(3) Let $F(a) = \mathbb{E}(e^{a \cdot X - |X|^2})$. Show that F is finite-valued and smooth.
(4) We aim now to show that there exists $a^* \in \mathbb{R}^d$ which minimizes F. Assuming this for the moment, show that $Z = e^{a^* \cdot X - |X|^2}$ satisfies the conclusions of the statement.
(5) Now suppose for the sake of finding a contradiction that F does not achieve its minimum. Let $(a_n)_n$ be such that $F(a_n) \downarrow \inf_a F(a)$. Why can we assume that $(a_n)_n$ is unbounded?
(6) Let $\hat{a}_n = a_n/|a_n|$. Why does the sequence $(\hat{a}_n)$ have a convergent subsequence?
(7) Suppose $|a_n| \to \infty$ and $\hat{a}_n \to \alpha$. Show that $\mathbb{P}(\alpha \cdot X > 0) = 0$.
(8) Use the hypothesis of the statement to show $\alpha \cdot X = 0$ a.s. Why is this our desired contradiction?

Solution 9. (1) Let $X_i = \tilde{S}_1^i - \tilde{S}_0^i$ be the change in price of the i-th asset, discounted by the price of asset 0. Then the elements Z of $\mathcal{Z}$ such that $\mathbb{E}(Z) = 1$ are the densities of equivalent martingale measures.
(2) If the X_i are not linearly independent, then one can pass to a linearly independent subset $(X_{i_k})_k$ with the same span. Suppose $a \cdot X \geq 0$ a.s. implies $a \cdot X = 0$ a.s. By taking $a_i = 0$ unless $i = i_k$ for some k, we see that this subset satisfies the same hypothesis.
Also, if there exists a random variable Z such that $\mathbb{E}(Z X_{i_k}) = 0$ for all k, then $\mathbb{E}(X_i) = 0$ for all i since each X_i can be written as a linear combination of the X_{i_k}. Therefore, it is enough to prove the theorem for this linearly independent subset.
(3) By completing the square, we have $F(a) \leq e^{|a|^2/4}$. The smoothness of F follows from repeated application of the dominated convergence theorem. Indeed, it is well-known that everywhere finite moment generating functions are smooth.
(4) If F is minimized at a^*, then the first order condition

$$\nabla F(a^*) = \mathbb{E}[X e^{a^* \cdot X - |X|^2}] = 0$$

must hold. Since $Z = e^{a^* \cdot X - |X|^2} > 0$ a.s., we are done.
(5) If the sequence $(a_n)_n$ were bounded, then there would exist a subsequence $(a_{n_k})_k$ such that $a_{n_k} \to a^*$ for some $a^* \in \mathbb{R}^d$. But by the smoothness of F, we have the limit $F(a_{n_k}) \to F(a^*)$. But this limit is the infimum of $F(a)$ over $a \in \mathbb{R}^d$, implying that a^* is the minimizer, contradicting our assumption that the minimum is not attained.
(6) The sequence $(\hat{a}_n)_n$ is bounded, so this is another application of the Bozano–Weierstrass theorem. Notice that the limit α is on the unit sphere.
(7) Note that $e^{a_n \cdot X} = (e^{\hat{a}_n \cdot X})^{|a_n|} \to \infty$ on $\{\alpha \cdot X > 0\}$. Since

$$\mathbb{E}(\liminf_{n\to\infty} e^{a_n \cdot X - |X|^2}) \leq \lim_{n\to\infty} F(a_n) = \inf_{a \in \mathbb{R}^d} F(a) < \infty$$

by Fatou's lemma, we must have $\mathbb{P}(\alpha \cdot X > 0) = 0$.
(8) By the last part we have $-a \cdot X \geq 0$ a.s. But by hypothesis, we have $\alpha \cdot X = 0$ a.s. Since we've assumed that the X_i are linearly independent, we must have $\alpha = 0$. But this contradicts the fact that $|\alpha| = 1$ by part (6). Hence the statement has been established.

Problem 10. Let $X_1, \dots, X_d$ and Y random variables, and let

$$\mathcal{Z} = \{Z > 0 \text{ a.s.} : \mathbb{E}(Z|X_i|) < \infty \text{ and } \mathbb{E}(ZX_i) = 0 \text{ for all } i\} \neq \emptyset.$$

We aim to show that if $\mathbb{E}(ZY) = 0$ for all $Z \in \mathcal{Z}$ such that $\mathbb{E}(Z|Y|) < \infty$, then

$$Y = a \cdot X \text{ a.s. for some } a \in \mathbb{R}^d.$$

(1) Why does this statement imply the hard direction of the characterization of attainable claims?
(2) How can we modify the argument in the previous problem to find random variables $Z_r \in \mathcal{Z}$ of the form

$$Z_r = e^{a_r \cdot X + rY - |X|^2 - Y^2}$$

for $r \in \{0, 1\}$.
(3) Show that $Y = b \cdot X + \log(Z_1/Z_0)$ for some $b \in \mathbb{R}^d$.
(4) Use Jensen's inequality to show

$$\mathbb{E}\left[Z_0 \log \frac{Z_1}{Z_0}\right] \leq \mathbb{E}(Z_0) \log \frac{\mathbb{E}(Z_1)}{\mathbb{E}(Z_0)}$$

and deduce $\mathbb{E}(Z_1) \geq \mathbb{E}(Z_0)$.
(5) Find a similar argument to show $\mathbb{E}(Z_0) \geq \mathbb{E}(Z_1)$.
(6) Hence $\mathbb{E}(Z_1) = \mathbb{E}(Z_0)$. Use the strict concavity of the logarithm function to show that $Z_1 = Z_0$ a.s. Why does this imply the conclusion of the statement?

Solution 10. (1) Let $X_i = \tilde{S}_1^i - \tilde{S}_0^i$ be the change in discounted price of the i-th asset and identify the elements of $\mathcal{Z}$ with densities of equivalent martingale measures, as described above.
(2) Just redefine F to be

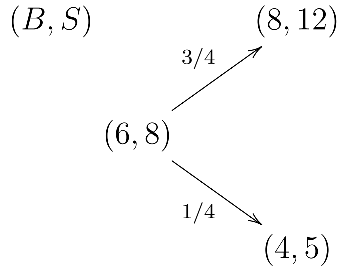
$$F_r(a) = \mathbb{E}[e^{a \cdot X + rY - |X|^2 - Y^2}]$$

for $r \in \{0, 1\}$, and let a_r be the minimizer of F_r.
(3) $b = a_0 - a_1$.
(4) Since $\mathbb{E}[Z_0 Y] = 0$ and $\mathbb{E}[X] = 0$, Jensen's inequality implies

$$0 = \mathbb{E}[Z_0(Y - b \cdot X)] = \mathbb{E}\left[Z_0 \log \frac{Z_1}{Z_0}\right] \leq \mathbb{E}(Z_0) \log \frac{\mathbb{E}(Z_1)}{\mathbb{E}(Z_0)}.$$

(5) This time bound $\mathbb{E}\left[Z_1 \log \frac{Z_1}{Z_0}\right] = -\mathbb{E}\left[Z_1 \log \frac{Z_0}{Z_1}\right]$ by Jensen's inequality.
(6) Since log is strictly concave, Jensen's inequality holds with equality if and only if the random variable $\frac{Z_1}{Z_0} = C$ for a positive constant. But we have just deduced $\mathbb{E}[Z_0 \log C] \leq 0$ so that $C \leq 1$, and $\mathbb{E}[Z_1 \log C] \geq 0$ so that $C \geq 1$. Hence $C = 1$, and $Y = b \cdot X$ as claimed.

Problem 11. Consider a the binomial two-asset model with prices



Introduce a call option with strike 10, with payout $\xi_1 = (S_1 - 10)^+$. Find its unique no-arbitrage price ξ_0. What is the replicating portfolio for this option?

Solution 11. There are two ways to approach this. It is important that you understand both.

A replicating portfolio for the option can be found by solving

$$\begin{aligned} 8\phi + 12\pi &= 2 = (12-10)^+ \\ 4\phi + 5\pi &= 0 = (5-10)^+ \end{aligned}$$

with solution $\phi = -5/4$ and $\pi = 1$. The time-0 value of this portfolio is $(-5/4)(6)+(1)(8) = 1/2$. Hence, there is no arbitrage if and only if the time-0 price of the option is the cost of replication $\xi_0 = 1/2$.

The other way is to find the equivalent martingale measure $\mathbb{Q}$ which solves $\mathbb{E}^{\mathbb{Q}}(\tilde{S}_1) = \tilde{S}_0$; that is,

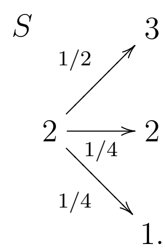
$$q12/8 + (1-q)5/4 = 8/6$$

or $q = 1/3$. There is no arbitrage iff

$$\xi_0 = \mathbb{E}^{\mathbb{Q}}(\xi_1 B_0/B_1) = (1/3)(2)(6/8) + (2/3)(0)(6/4) = 1/2$$

as before.

Problem 12. Consider a trinomial two-asset model with $B_0 = B_1 = 1$ and S given by



Find all equivalent martingale measures for this model. Now introduce a call option with strike 2, with payout $\xi_1 = (S_1 - 2)^+$. Show directly that the payout ξ_1 cannot be replicated by trading in the stock and bond. Prove that there is no arbitrage if and only if $0 < \xi_0 < \frac{1}{2}$.

Find the attainable random variable $X^* = x^* + \pi^* \cdot (S_1 - S_0)$ which minimizes expected square hedging error $\mathbb{E}[(X - \xi_1)^2]$. Find the random variable ρ^* that minimizes $\rho \mapsto \mathbb{E}[\rho^2]$ subject to $\mathbb{E}[\rho(B_1, S_1)] = (B_0, S_0)$, and verify $x^* = \mathbb{E}(Z^* \xi_1)$.

Solution 12. An equivalent martingale measure solves $\mathbb{E}^{\mathbb{Q}}(S_1) = S_0$, (no discounting is needed since asset 0 is cash)

$$\begin{aligned} 3p + 2q + r &= 2 \\ p + q + r &= 1 \end{aligned}$$

and hence $(p, q, r) = (p, 1 - 2p, p)$ for $0 < p < 1/2$.

Denoting by ϕ and π the holding in the bond and stock that would replicate the payout ξ_1, we need the three conditions to hold simultaneously:

$$\begin{aligned} \phi + 3\pi &= 1 \\ \phi + 2\pi &= 0 \\ \phi + \pi &= 0 \end{aligned}$$

Impossible!

There will be no arbitrage iff $\xi_0 = \mathbb{E}^{\mathbb{Q}}(\xi_1)$ for some equivalent martingale measure. Evaluating the expectation in term of $\mathbb{Q}$ introduced earlier gives that there is no arbitrage iff $\xi_0 = p$ and we have already proved that no-arbitrage was equivalent to $p \in (0, \frac{1}{2})$.

We must now minimize

$$\frac{1}{2}(x + \pi - 1)^2 + \frac{1}{4}x^2 + \frac{1}{4}(x - \pi)^2$$

to find $x^* = 4/11$ and $\pi^* = 6/11$.

Now we must find ρ^* that minimizes $\rho \mapsto \mathbb{E}[\rho^2]$ subject to $\mathbb{E}[\rho(B_1, S_1)] = (B_0, S_0)$. We have seen that all such ρ are of the form

$$\rho(\omega) = \begin{cases} 2p & \text{on } A_3 \\ 4(1 - 2p) & \text{on } A_2 \\ 4p & \text{on } A_1 \end{cases}$$

where $A_i = \{S_1 = i\}$. Hence, we must minimize

$$\frac{1}{2}(2p)^2 + \frac{1}{4}[4(1 - 2p)]^2 + \frac{1}{4}(4p)^2$$

to get $p^* = 4/11$. Note $\mathbb{E}(\rho^* \xi_1) = (4/11)(1) + (3/11)(0) + (4/11)(0) = 4/11 = x^*$

Problem 13. Let $\mathcal{Z}$ be the set of Radon–Nikodym derivatives of equivalent martingale measures for a given market model. Prove that $\mathcal{Z}$ is a convex subset of L^1. Show by example that $\mathcal{Z}$ is *not* necessarily closed in L^1.

(Remember that a random variable Z is in L^1 iff $\mathbb{E}(|Z|) < \infty$, and that a subset $\mathcal{Z}$ of L^1 is closed in L^1 iff for every sequence $Z_1, Z_2, \ldots$ of random variables in $\mathcal{Z}$ such that $\mathbb{E}(|Z_n - Z|) \to 0$ for some random variable $Z \in L^1$, the limit point Z is also in $\mathcal{Z}$.)

Solution 13. If Z_1 and Z_2 are in $\mathcal{Z}$ and $\theta \in [0, 1]$ then

- $\theta Z_0 + (1 - \theta) Z_1 > 0$ a.s., since $Z_1 > 0$ a.s. and $Z_2 > 0$ a.s.,
- $\mathbb{E}[\theta Z_0 + (1 - \theta) Z_1] = \theta \mathbb{E}[Z_0] + (1 - \theta)\mathbb{E}[Z_1] = 1$,
- $\mathbb{E}\{[\theta Z_0 + (1 - \theta) Z_1](\tilde{S}_1 - \tilde{S}_0) = \theta \mathbb{E}[Z_0(\tilde{S}_1 - \tilde{S}_0)] + (1 - \theta)\mathbb{E}[Z_1(\tilde{S}_1 - \tilde{S}_0)] = 0$

and hence $\theta Z_0 + (1 - \theta) Z_1 \in \mathcal{Z}$.

Now consider the market model from the previous problem, and let $A_i = \{S_1 = i\}$, so that $\mathbb{P}(A_1) = 1/4 = \mathbb{P}(A_2)$ and $\mathbb{P}(A_3) = 1/2$. Now let

$$Z_n = \begin{cases} 2/n & \text{on } A_3 \\ 4(1 - 2/n) & \text{on } A_2 \\ 4/n & \text{on } A_1 \end{cases}$$

Note that Z_n is the density $\frac{d\mathbb{Q}_n}{d\mathbb{P}}$ of the equivalent martingale measure which assigns $\mathbb{Q}_n(A_3) = 1/n = \mathbb{Q}_n(A_1)$ and $\mathbb{Q}_n(A_2) = 1 - 2/n$. Now $Z_n \to Z$ in L^1, where

$$Z_n = \begin{cases} 0 & \text{on } A_3 \\ 4 & \text{on } A_2 \\ 0 & \text{on } A_1 \end{cases}$$

since $\mathbb{E}(|Z_n - Z|) = (2/n)(1/2) + (8/n)(1/4) + (4/n)(1/4) = 4/n \to 0$. However, Z does not give rise to an equivalent measure since $Z = 0$ with positive probability. (The limit measure is absolutely continuous with respect to $\mathbb{P}$, however.)

Problem 14. Now consider a two-asset model (B, S), where asset 0 is a risk-free bond with risk-free (not random) rate of return r, given by

$$r = \frac{B_1}{B_0} - 1,$$

and where asset 1 is a stock with expected rate of return R given by

$$R = \frac{\mathbb{E}(S_1)}{S_0} - 1.$$

(Assume S_1 is integrable.) Given his initial wealth X_0, an investor maximizes the expected utility $\mathbb{E}[U(X_1)]$ for a given increasing, concave utility function U. Assume that the maximum is attained when he holds π^* shares of the stock. Show that if $\pi^* > 0$, then $R \geq r$. Conversely, show that if $R > r$ then $\pi^* \geq 0$. Do these results agree with your intuition?

(This is harder: Suppose that U is strictly concave and that S_1 is not constant. Show that if $\pi^* > 0$ then $R > r$. Can you find an example where $R > r$, but $\pi^* = 0$?)

Solution 14. If the initial wealth is X_0, then by the self-financing condition, the wealth at time 1 as a function of π is

$$X_1^{(\pi)} = (1 + r)X_0 + \pi[S_1 - (1 + r)S_0].$$

Let π^* be the choice of $\pi \in \mathbb{R}$ that maximizes $\mathbb{E}[U(X_1^{(\pi)})]$. On the one hand, we have

$$U(X_1^{(0)}) \leq \mathbb{E}[U(X_1^{(\pi^*)})]$$

by the optimality of $\pi = \pi^*$ as compared to $\pi = 0$, and since $X_1^{(0)} = (1+r)X_0$ is not random by assumption.

On the other hand, we have

$$\mathbb{E}[U(X_1^{(\pi^*)})] \leq U(\mathbb{E}[X_1^{(\pi^*)}])$$

by Jensen's inequality. Since U is increasing, we have

$$X_1^{(0)} \leq \mathbb{E}(X_1^{(\pi^*)}) = X_1^{(0)} + \pi^* \mathbb{E}[S_1 - (1 + r)S_0]$$

and the result follows.

Now if U is strictly concave, and S_1 is not constant, then the above inequality are strict. For the example where $R > r$ but $\pi^* = 0$, just choose the distribution of S_1 such that $\mathbb{E}[U(X_1^{(\pi)})] = -\infty$ unless $\pi = 0$. For instance, let $U(x) = -e^{-x}$, let $r = 0$ and $S_1 - S_0$ have density

$$f(s) = \frac{1}{(|s - 1| + 1)^3}$$

In this example, the investor is extremely risk adverse: holding any non-zero number of shares is so risky it yields negative infinite expected utility.

Another way of looking at this problem, but harder to justify rigorously, is to consider the function $\pi \mapsto \mathbb{E}[U(X_1^{(\pi)})]$. Since this function is maximized at π^*, the gradient

$$\mathbb{E}\{U'(X_1^{(\pi)})[S_1 - (1+r)S_0]\}$$

is non-negative if and only if $\pi \leq \pi^*$. Letting $\pi = 0$ above and noting that $U'(X_1^{(0)})$ is positive and non-random completes the argument.