Advanced Financial Models Michael Tehranchi

Example sheet 1 - Michaelmas 2019

Problem 1. Consider a market with $n = 1$ asset with prices $(P_t)_{t\geq 0}$ and dividends $(\delta_t)_{t\geq 1}$. Suppose there is no arbitrage.

(a) Show that there exists a strictly positive adapted process Y such that

$$P_t = \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{T} Y_u \delta_u | \mathcal{F}_t\right) + \frac{1}{Y_t}\mathbb{E}(Y_T P_T | \mathcal{F}_t)$$

for all $0 \leq t \leq T$.

(b) Now suppose that $P_t \geq 0$ and $\delta_t \geq 0$ almost surely for all t. Letting Y be the process in part (a), show that

$$P_t \geq \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{\infty} Y_u \delta_u | \mathcal{F}_t\right)$$

for all $t \geq 0$. Find a condition on P, Y and δ such that there is equality in the above inequality. [The right-hand side of the inequality could be thought of the fundamental value of the asset - i.e. the present discounted value of the stream of dividend payments. When there is strict inequality, the price of the asset is strictly greater than its fundamental value, modelling a price bubble. Note that no arbitrage does not forbid such price bubbles.]

Solution 1. (a) This is just the first fundamental theorem of asset pricing: Assuming no arbitrage there exists a martingale deflator Y meaning

$$P_t = \frac{1}{Y_t}\mathbb{E}[Y_{t+1}(P_{t+1} + \delta_{t+1})|\mathcal{F}_t].$$

In particular, the process

$$M_t = Y_t P_t + \sum_{s=1}^{t} Y_s \delta_s$$

is a martingale. The identity amounts to $\mathbb{E}(M_T - M_t|\mathcal{F}_t) = 0$.

(b) From part (a) with $t = 0$, we have

$$\mathbb{E}\left(\sum_{u=1}^{T} Y_u \delta_u | \mathcal{F}_t\right) \leq Y_0 P_0$$

since $P_T \geq 0$. Taking $T \to \infty$ by the monotone convergence theorem shows that $\sum_{u=1}^{\infty} Y_u \delta_u$ is finite-valued and integrable.

Since $P_T \geq 0$ we have

$$P_t \geq \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{T} Y_u \delta_u | \mathcal{F}_t\right) \to \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{\infty} Y_u \delta_u | \mathcal{F}_t\right)$$

by the conditional monotone convergence theorem, where the convergence is a.s. and in L^1.

Finally, note that we have

$$P_t - \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=t+1}^{\infty} Y_u \delta_u | \mathcal{F}_t\right) = \frac{1}{Y_t}\mathbb{E}(Y_T P_T | \mathcal{F}_t) - \frac{1}{Y_t}\mathbb{E}\left(\sum_{u=T+1}^{\infty} Y_u \delta_u | \mathcal{F}_t\right)$$

The second term on the right-hand side converges to 0 in L^1. Hence, the necessary and sufficient condition for equality is

$$\mathbb{E}(Y_T P_T|\mathcal{F}_t) \to 0$$

in L^1, which is equivalent to $Y_t P_t \to 0$ in L^1 by the tower property.

Problem 2. Consider an arbitrage-free market with $n = 1$ asset with prices $(P_t)_{t\geq 0}$ and dividends $(\delta_t)_{t\geq 1}$. Suppose that $\delta_t \geq 0$ almost surely for all $t \geq 1$, and that there exists a non-random time $T > 0$ such that $P_T \geq 0$ almost surely and $\mathbb{P}(P_T > 0) > 0$. Show that $P_0 > 0$. Must it be the case that $P_t > 0$ almost surely for all $0 \leq t < T$?

Solution 2. A 'primal' solution. Consider the strategy $H_t = 1$ for $1 \leq t \leq T$ and $H_{T+1} = 0$. Let the initial wealth be $x = 0$. The associated consumption process is $c_0 = -P_0$ and $c_t = \delta_t$ for $1 \leq t \leq T-1$ and $c_T = P_T + \delta_T$. By assumption, $c_t \geq 0$ a.s. for $1 \leq t \leq T$ and $\mathbb{P}(c_T > 0) > 0$. So if there is no arbitrage, then $c_0 < 0$, yielding the conclusion.

A 'dual' solution. By no arbitrage there exists a martingale deflator Y so that

$$P_0 = \frac{1}{Y_0}\mathbb{E}\left(\sum_{u=1}^{T} Y_u \delta_u\right) + \frac{1}{Y_0}\mathbb{E}(Y_T P_T).$$

(Note that we're using the convention that $\mathcal{F}_0$ is trivial.) The first term on the right-hand side is non-negative, and the second term is strictly positive, yielding the conclusion.

For the final part, no, it is possible for $\mathbb{P}(P_t = 0) > 0$ for some $0 < t < T$. For instance, let $(\xi_i)_i$ be i.i.d. with $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$. Let $S_0 = 1$ and $S_t = S_{t-1} + \xi_t$. Finally, $\tau = \inf\{t \geq 0 : S_t = 0\}$. Consider a market with $\delta_t = 0$ for all t and $P_t = S_{t\wedge\tau}$. Note there is no arbitrage since P is a martingale (so the constant process $Y = 1$ is a martingale deflator). Also $P_t \geq 0$ a.s. and $\mathbb{P}(P_T > 0) = \mathbb{P}(\tau > T) > 0$. Nevertheless, $\mathbb{P}(P_t = 0) = \mathbb{P}(\tau \leq t) > 0$ for $t \geq 1$.

Problem 3. Given a filtration $(\mathcal{F}_t)_{t\geq 0}$ let $(Z_t)_{t\geq 1}$ be a non-negative adapted integrable process such that $\mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1$ for all $t \geq 1$. Let

$$M_t = \prod_{s=1}^{t} Z_s.$$

Show that M is a martingale. [Hint: First show that M is a local martingale.]

Solution 3. The problem here is that in this course, conditional expectations are only defined for integrable random variables and we don't know a priori that M is integrable. But we do know that M is non-negative. This leads to two possible solutions.

Following the hint: Let $X_t = \sum_{s=1}^{t} Z_s - t$. Note that X is integrable and adapted, and that

$$\mathbb{E}[X_t - X_{t-1}|\mathcal{F}_{t-1}] = \mathbb{E}[Z_t - 1|\mathcal{F}_{t-1}] = 0.$$

Hence X is a martingale. Finally, note that

$$M_t - M_{t-1} = M_{t-1}(Z_t - 1) = M_{t-1}(X_t - X_{t-1}).$$

This shows that M is a local martingale, being the martingale transform of the previsible process $M_{\cdot-1}$ with respect to the martingale X. From lectures, non-negative local martingales in DISCRETE TIME are true martingales, so we are done.

Another solution is to let $Z_t^n = Z_t \wedge n$. Since $Z_t^n \le Z_t$ we have $\mathbb{E}(Z_t^n|\mathcal{F}_{t-1}) \le 1$. Now let $M_t^n = \prod_{s=1}^t Z_s^n$. Note that both M_t^n is integrable (being bounded) and that

$$\mathbb{E}(M_t^n|\mathcal{F}_{t-1}) = M_{t-1}^n \mathbb{E}(Z_t^n|\mathcal{F}_{t-1}) \le M_{t-1}^n$$

by the slot property. Thus M^n is a supermartingale and $\mathbb{E}(M_t^n) \le M_0^n = 1$. Since $0 \le M_t^n \uparrow M_t$ the monotone convergence theorem says $\mathbb{E}(M_t) \le 1$. In particular, M_t is integrable. So we can use the slot property once more:

$$\mathbb{E}(M_t|\mathcal{F}_{t-1}) = M_{t-1}\mathbb{E}(Z_t|\mathcal{F}_{t-1}) = M_{t-1}$$

as desired.

Problem 4. Consider a discrete time model with a single asset with positive prices $(P_t)_{t\ge0}$ and non-negative dividends $(\delta_t)_{t\ge1}$. Show that there is a self-financing (pure-investment) trading strategy with corresponding wealth process

$$Q_t = P_t \prod_{s=1}^t \left(1 + \frac{\delta_s}{P_s}\right).$$

Let Y be a positive adapted process. Show that

$$M_t = Y_t P_t + \sum_{s=1}^t Y_s \delta_s$$

defines a martingale if and only if

$$N_t = Y_t Q_t$$

defines a martingale.

Solution 4. Let

$$H_t = \frac{Q_{t-1}}{P_{t-1}}$$

for $t \ge 1$. Notice that H is previsible and self-financing (without consumption), since

$$H_t(P_t + \delta_t) = Q_t = H_{t+1}P_t.$$

The strategy H consists of holding one share of the asset initially, and reinvesting the dividend payments.

Note the identity

$$Y_{t+1}Q_{t+1} - Y_tQ_t = H_{t+1}[Y_{t+1}(P_{t+1} + \delta_{t+1}) - Y_tP_t].$$

Letting

$$M_t = Y_t P_t + \sum_{s=1}^t Y_s \delta_s$$

we see that

$$Y_tQ_t = Y_0P_0 + \sum_{s=1}^t H_s(M_s - M_{s-1}).$$

We recognise YQ as the martingale transform of the predictable process H with respect to M. If M is a martingale, then YQ is a local martingale. But non-negative local martingales in discrete time are true martingales.

Similarly, writing

$$M_t = Y_0 P_0 + \sum_{s=1}^{t} \frac{1}{H_s}(Y_s Q_s - Y_{s-1} Q_{s-1})$$

we see that if YQ is a martingale, then M is martingale.

Problem 5. Consider a two-asset model with prices given by

$$\begin{tikzcd} (P^1, P^2) & & (3,9) \\ & (4,6) \arrow[ur, "1/4"] \arrow[r, "1/4"'] \arrow[dr, "1/2"'] & (6,8) \\ & & (6,4) \end{tikzcd}$$

Is there arbitrage in this market? If so, find all arbitrages. If not, find all pricing kernels.

Solution 5. This is a very important problem. To ensure you understand it, below are three solutions.

(1) Let $H \in \mathbb{R}^2$ be a candidate absolute arbitrage, such that $H \cdot P_0 \leq 0 \leq H \cdot P_1$. We will show $H \cdot P_0 = 0 = H \cdot P_1$ and hence there is no arbitrage.

Label the outcomes $\Omega = \{A, B, C\}$ where $P_1(A) = (3, 9)$, etc. and let $H = (h, k)$. The inequalities become

(0): $4h + 6k \leq 0$
(A): $3h + 9k \geq 0$
(B): $6h + 8k \geq 0$
(C): $6h + 4k \geq 0$

By subtracting appropriate multiples of inequality (0) from the others to eliminate h, we have

(A'): $(9 - \frac{3}{4} \times 6)k = 9k/2 \geq 0$
(B'): $(8 - \frac{6}{4} \times 6)k = -k \geq 0$
(C'): $(4 - \frac{6}{4} \times 6)k = -5k \geq 0$

Inequalities (A') and (B') together imply $k = 0$. Plugging this into (0) and (A) above imply $h = 0$, as desired.

(2) Since this market has a numéraire asset (in fact, both assets have strictly positive prices), there is no arbitrage if and only if there is no terminal consumption arbitrage. So, as above, let H be a candidate terminal consumption arbitrage with $H \cdot P_0 = 0$ and $H \cdot P_1 \geq 0$. We will show $H \cdot P_1 = 0$. Labelling the outcomes as before, we have

(0): $4h + 6k = 0$
(A): $3h + 9k \geq 0$
(B): $6h + 8k \geq 0$
(C): $6h + 4k \geq 0$

We then proceed as above.

(3) By the first fundamental theorem of asset pricing there is no arbitrage if and only if there is a pricing kernel. So we seek a positive random variable ρ such that $\mathbb{E}(\rho P_1) = P_0$. Letting

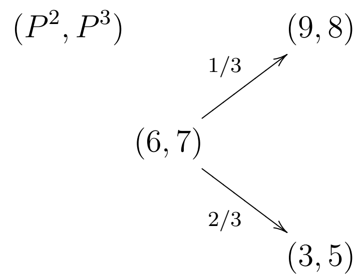
$\rho(A) = a, \rho(B) = b$ and $\rho(C) = c$ we have

$$\frac{1}{4}(3a) + \frac{1}{4}(6b) + \frac{1}{2}(6c) = 4$$
$$\frac{1}{4}(9a) + \frac{1}{4}(8b) + \frac{1}{2}(4c) = 6$$

All positive solutions of these two equations in three unknowns can be written as

$$(a, b, c) = u\left(\frac{40}{21}, 0, \frac{6}{7}\right) + (1-u)\left(\frac{8}{15}, \frac{12}{5}, 0\right), \quad 0 < u < 1.$$

Problem 6. Consider a three-asset model with asset 1 cash so that $P_0^1 = P_1^1 = 1$ and assets 2 and 3 given by



Is there arbitrage in this market? If so, find all arbitrages. If not, find all pricing kernels.

Solution 6. Yes, there is an arbitrage. A candidate arbitrage $H = (g, h, k)$ would satisfy

(0): $g + 6h + 7k \le 0$
(A): $g + 9h + 8k \ge 0$
(B): $g + 3h + 5k \ge 0$

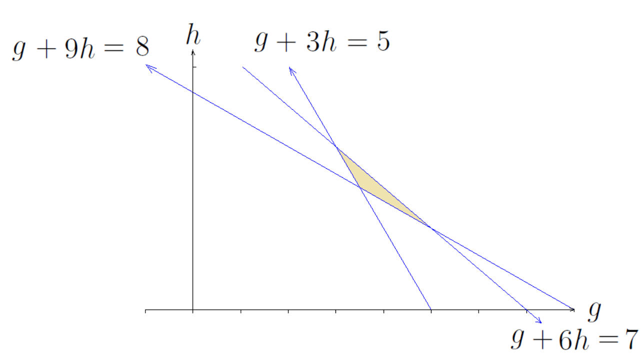
Adding (A) and (B) and subtracting twice (0) yields

$$-k \ge 0.$$

So fix such a k, for instance $k = -1$ and plot the inequalities in the variables g and h.

$$g + 6h \le 7$$
$$g + 9h \ge 8$$
$$g + 3h \ge 5$$

See the figure. All solutions are convex combinations of the extreme points



$$(g,h) = (5,1/3); (7/2,1/2); (3,2/3)$$

Hence, all arbitrages are of the form

$$(g,h,k) = u(15,1,-3) + v(9,2,-3) + w(7,1,-2), \quad \min\{u,v,w\} \geq 0 \text{ and } \max\{u,v,w\} > 0.$$

By the 1FTAP, there are no pricing kernels. But let's see this directly in this example: Let $\Omega = \{A, B\}$ and let ρ be a candidate pricing kernel with $\rho(A) = a, \rho(B) = b$. Then we have three equations and two unknowns:

$$\begin{aligned}
(\tfrac{1}{3})a + (\tfrac{2}{3})b &= 1 \\
(\tfrac{1}{3})9a + (\tfrac{2}{3})3b &= 6 \\
(\tfrac{1}{3})8a + (\tfrac{2}{3})5b &= 7.
\end{aligned}$$

The above system has no solution.

Problem 7. * Let X be a martingale, K a previsible process, and M_0 a constant. Let

$$M_t = M_0 + \sum_{s=1}^{t} K_s(X_s - X_{s-1}).$$

Show that if M_T is integrable for some non-random time $T > 0$, then $(M_t)_{0 \leq t \leq T}$ is a true martingale. Hint: Show that M_{T-1} is integrable.

Solution 7. Here are two solutions. The first uses the same trick as the lectures. Let $\tau_N = \inf\{t \geq 0 : |K_{t+1}| > N\}$. Note $M_s \mathbb{1}_{\{t \leq \tau_N\}}$ is integrable for all $0 \leq s \leq t$, since X is integrable by definition of martingale, and K_s is bounded on $\{t \leq \tau_N\}$. Hence we have

$$\begin{aligned}
\mathbb{E}[M_T \mathbb{1}_{\{\tau_N \leq T\}} | \mathcal{F}_{T-1}] &= \mathbb{E}[M_{T-1}\mathbb{1}_{\{T \leq \tau_N\}} + K_T \mathbb{1}_{\{T \leq \tau_N\}}(X_T - X_{T-1}) | \mathcal{F}_{T-1}] \\
&= M_{T-1}\mathbb{1}_{\{T \leq \tau_N\}} + K_T \mathbb{1}_{\{T \leq \tau_N\}} \mathbb{E}[X_T - X_{T-1} | \mathcal{F}_{T-1}] \\
&= M_{T-1}\mathbb{1}_{\{T \leq \tau_N\}}.
\end{aligned}$$

Now, we have assumed that M_T is integrable, and since $|M_T \mathbb{1}_{\{\tau_N \leq T\}}| \leq |M_T|$ we can apply the dominated convergence theorem

$$\begin{aligned}
\mathbb{E}[M_T | \mathcal{F}_{T-1}] &= \mathbb{E}[\lim_N M_T \mathbb{1}_{\{\tau_N \leq T\}} | \mathcal{F}_{T-1}] \\
&= \lim_N M_{T-1} \mathbb{1}_{\{T \leq \tau_N\}} \\
&= M_{T-1}.
\end{aligned}$$

This shows that M_{T-1} is integrable, and by induction $(M_t)_{0 \leq t \leq T}$ is integrable. An integrable local martingale in discrete time is a true martingale.

A second solution is similar. Let $A_N = \{|K_T| \leq N\}$ and note that $K_T \mathbb{1}_{A_N}$ is bounded and $\mathcal{F}_{T-1}$-measurable. Also $M_{T-1}\mathbb{1}_{A_N}$ is integrable, since

$$M_{T-1}\mathbb{1}_{A_N} = M_T \mathbb{1}_{A_N} - K_T \mathbb{1}_{A_N}(X_T - X_{T-1})$$

Hence

$$\begin{aligned}
\mathbb{E}[M_T \mathbb{1}_{A_N} | \mathcal{F}_{T-1}] &= \mathbb{E}[M_{T-1}\mathbb{1}_{A_N} + K_T \mathbb{1}_{A_N}(X_T - X_{T-1}) | \mathcal{F}_{T-1}] \\
&= M_{T-1}\mathbb{1}_{A_N} + K_T \mathbb{1}_{A_N} \mathbb{E}[X_T - X_{T-1} | \mathcal{F}_{T-1}] \\
&= M_{T-1}\mathbb{1}_{A_N}.
\end{aligned}$$

Send $N \to \infty$ as before.

Problem 8. This problem leads you through an alternative proof of the 1FTAP using the following version of the separating hyperplane theorem: Let $C \subset \mathbb{R}^n$ be convex and $x \in \mathbb{R}^n$ not contained in C. Then there exists a $\lambda \in \mathbb{R}^n$ such that $\lambda \cdot (y - x) \geq 0$ for all $y \in C$, where the inequality is strict for at least one $y \in C$.

We are given a market model $(P_t)_{t\in\{0,1\}}$.

(a) Define a collection of random variables by

$$\mathcal{Z} = \{Z : Z > 0 \text{ a.s. and } \mathbb{E}(Z\|P_1\|) < \infty\}.$$

Show that $\mathcal{Z}$ not empty and convex.

(b) Now define a subset of $\mathbb{R}^n$ by

$$\mathcal{P} = \{\mathbb{E}(ZP_1) : Z \in \mathcal{Z}\}.$$

Show that $\mathcal{P}$ is not empty and convex. Furthermore, show that if $P_0 \in \mathcal{P}$ there exists a martingale deflator $(Y_t)_{t\in\{0,1\}}$.

For the rest of the problem assume $P_0 \notin \mathcal{P}$. We must find an arbitrage.

(c) Use the given separating hyperplane theorem to show that there exists a vector $H \in \mathbb{R}^n$ such that $\mathbb{E}(ZH \cdot P_1) \geq H \cdot P_0$ for all $Z \in \mathcal{Z}$ with strict inequality for all at least one $Z \in \mathcal{Z}$.

(d) Use the conclusion of part (c) to show $H \cdot P_0 \leq 0$. [Hint: fix an element $Z_0 \in \mathcal{Z}$, and let $Z = \varepsilon Z_0$. Now look at the inequality when $\varepsilon \downarrow 0$.]

(e) Let $A = \{H \cdot P_1 < 0\}$. By setting $Z = (\frac{1}{\varepsilon}\mathbb{1}_A + 1)Z_0$, show $\mathbb{P}(A) = 0$.

(f) Finally, by appealing to the conclusion of part (c), show that H is an arbitrage.

Solution 8. (a) The set $\mathcal{Z}$ is not empty since $Z_0 = e^{-\|P_1\|}$ is an element. Also if $Z_0, Z_1 \in \mathcal{Z}$ and $Z_\theta = (1-\theta)Z_0 + \theta Z_1$ for some $0 \leq \theta \leq 1$ then $Z_\theta > 0$ and

$$\mathbb{E}(Z_\theta\|P_1\|) = (1-\theta)\mathbb{E}(Z_0\|P_1\|) + \theta\mathbb{E}(Z_1\|P_1\|) < \infty.$$

So $\mathcal{Z}$ is convex.

(b) The non-emptiness and convexity of $\mathcal{P}$ follow from (a). If $P_0 \in \mathcal{P}$ then $\mathbb{E}(ZP_1) = P_0$ for some $Z > 0$. Hence $Y_1 = Z$, $Y_0 = 1$ is a martingale deflator.

(c) The separating hyperplane theorem says there is vector $H \in \mathbb{R}^n$ such that $H\cdot(p-P_0) \geq 0$ for all $p \in \mathcal{Z}$ and $H \cdot (p - P_0) > 0$ for at least one $p \in \mathcal{P}$. The required conclusion follows from noting that every element $p \in \mathcal{P}$ is of the form $p = \mathbb{E}(ZP_1)$ for some $Z \in \mathcal{Z}$.

(d) Let $Z = \varepsilon Z_0$ as hinted. From (c) we have $H \cdot P_0 \leq \varepsilon\mathbb{E}(Z_0 H \cdot P_1)$. Now send $\varepsilon \downarrow 0$.

(e) From (c) we have

$$\mathbb{E}(Z_0 H \cdot P_0 \mathbb{1}_A) \geq \varepsilon[H \cdot P_0 - \mathbb{E}(Z_0 H \cdot P_0)].$$

Sending $\varepsilon \downarrow 0$ yields $\mathbb{E}(Z_0 H \cdot P_0 \mathbb{1}_A) \geq 0$. But the integrand is non-positive almost surely. By the pigeon-hole principle, the integrand is zero almost surely. Since $Z_0 > 0$, we have $\mathbb{P}(A) = 0$.

(f) From (d) we have $H \cdot P_0 \leq 0$ and from (e) we have $H \cdot P_1 \geq 0$. But from (c) there exists a $Z \in \mathcal{Z}$ such that $\mathbb{E}(ZH \cdot P_1) > H \cdot P_0$. Hence $\mathbb{P}(H \cdot P_0 = 0 = H \cdot P_1) = 0$.

Problem 9. (Stiemke's theorem) Let A be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^\top y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^\top y \neq 0$.

What does this have to do with finance?

Solution 9. Let $\Omega = \{1, \ldots, n\}$ and $\mathbb{P}(\{j\}) = 1/n$ for all $j \in \Omega$. Let $P_0 = 0$, and define a random vector by $P_1 : \Omega \to \mathbb{R}^m$ by the $P_1^i(j) = a_{i,j}$ where $A = (a_{i,j})_{i,j}$. Consider a m-asset market model with prices P.

Since $H \cdot P_0 = 0$ for all H, an arbitrage is a vector H such that $H \cdot P_1 \geq 0$ a.s. and $\mathbb{P}(H \cdot P_1 > 0) > 0$. Or using the notation of the problem, an arbitrage is a vector $y = H \in \mathbb{R}^m$ such that with $(A^T y)_j \geq 0$ for all $j = 1, \ldots, n$ such that $A^T y \neq 0$.

A pricing kernel is a positive random variable ρ such that

$$0 = \mathbb{E}(\rho P_1) = \sum_j \frac{1}{n} \rho(j) P_1(j).$$

Or, letting $x_j = \rho(j)/n$ a pricing kernel is a vector $x \in \mathbb{R}^n$ such that $Ax = 0$ and $x_j > 0$ for all j.

The result conclusion is just an application of the 1FTAP.

We have now seen one proof of the 1FTAP in lecture and another in the previous problem. For the sake of developing intuition, here's yet another one: As before, the easy direction is to show that the existence of a pricing kernel implies no arbitrage: Suppose that there exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$. If there exists a $y \in \mathbb{R}^m$ with $(A^\top y)_i \geq 0$ for all $i = 1, \ldots, n$ then $x \cdot (A^\top y) = (Ax) \cdot y = 0$. Hence $A^\top y = 0$.

Now we prove the harder direction using a version of the separating hyperplane theorem. Suppose that there is no arbitrage: If there exists a $y \in \mathbb{R}^m$ with $(A^\top y)_i \geq 0$ for all $i = 1, \ldots, n$ then $A^\top y = 0$. Let

$$S = \{A^\top y : y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

and let

$$C = \left\{ u \in \mathbb{R}^n : u_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ and } \sum_{i=1}^n u_i = 1 \right\} \subset \mathbb{R}^n.$$

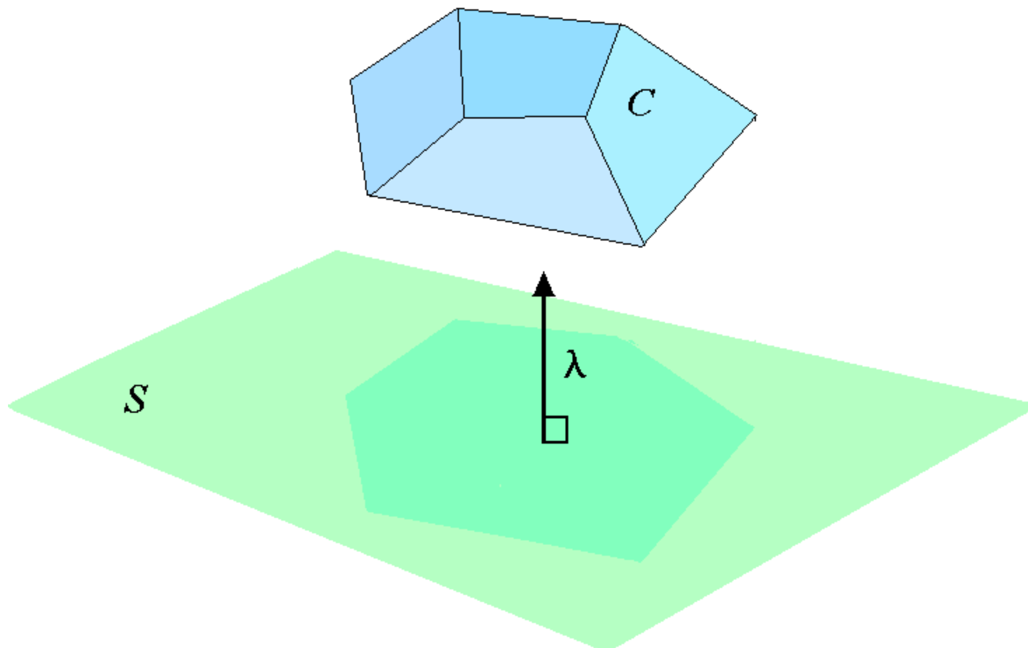
By assumption, the subspace S and the convex compact set C are disjoint. Indeed, if $v \in S$, then $v = A^T y$ for some y. If $v_i \geq 0$ for all i, then $v = 0$ by the no-arbitrage assumption. Hence $\sum_i v_i = 0 \neq 1$ and v is not in C.

The situation is illustrated by the figure. A version of the separating hyperplane theorem, says there exists a vector $\lambda \in \mathbb{R}^n$ such that

$$\lambda \cdot v = 0 \text{ for all } v \in S$$
$$\lambda \cdot u > 0 \text{ for all } u \in C.$$

(Try proving this!) We will be done once we show that $\lambda_j > 0$ for all j. So fix a $j \in \{1, \ldots, n\}$ and let $e \in \mathbb{R}^n$ be given by $e_j = 1$ and $e_i = 0$ for all $i \neq j$. Then $e \in C$ and $\lambda \cdot e = \lambda_j > 0$ as desired.

Problem 10. * Consider an arbitrage-free market with at least two assets, where no asset pays a dividend.
(a) Prove the *law of one price*: if $P_T^1 = P_T^2$ almost surely, where $T > 0$ is a non-random time, then $P_t^1 = P_t^2$ almost surely for all $0 \leq t \leq T$.
(b) Find an example of an arbitrage-free market for which there is a stopping time τ such that $P_\tau^1 = P_\tau^2$ almost surely, and yet $P_0^1 \neq P_0^2$.



Solution 10. There is a primal and a dual argument.

(a Primal) Suppose $P^1_T = P^2_T$ a.s. Let $\tau = \inf\{t \geq 0 : P^1_t \neq P^2_t\}$. Consider the portfolio $H_{t+1} = \mathbb{1}_{\{\tau \leq t\}}\text{sign}(P^1_\tau - P^2_\tau)(-1, 1)$ for $0 \leq t < T$ and $H_{T+1} = 0$. Note the corresponding consumption is $c_t \geq 0$ for all $0 \leq t \leq T$. Since

$$\{c_t > 0 \text{ for some } 0 \leq t \leq T\} = \{\tau \leq T\}$$

there is no arbitrage only if $\tau > T$ almost surely.

(a Dual) Suppose $P^1_T = P^2_T$ a.s. By 1FTAP there exists a martingale deflator Y Then

$$P^1_t = \frac{1}{Y_t}\mathbb{E}(Y_T P^1_T | \mathcal{F}_t) = \frac{1}{Y_t}\mathbb{E}(Y_T P^2_T | \mathcal{F}_t) = P^2_t.$$

Let $P^1_t = 1$ for all $t \geq 0$ and $P^2_t = \xi_1 + \ldots + \xi_t$ where the sequence $(\xi_n)_n$ is IID with $\mathbb{P}(\xi_n = \pm 1) = \frac{1}{2}$. This market has no arbitrage since the prices are martingales (so we may take the martingale deflator to be $Y_t = 1$ for all $t \geq 0$.) Now set $\tau = \inf\{t \geq 0 : P^2_t = 1\}$. Note $\tau < \infty$ almost surely, and $P^1_\tau = 1$ almost surely. However, $P^1_0 = 1 \neq 0 = P^2_0$, so the law of one price does not necessarily hold for unbounded stopping times.

Problem 11. (Tower property of conditional expectation) Let X and Y be identically distributed random variables taking values in the set $\{2^n : n \geq 0\}$ such that $X/Y \in \{1/2, 2\}$ almost surely and

$$\mathbb{P}(X = 2^n, Y = 2^{n+1}) = \frac{1}{4}2^{-n} = \mathbb{P}(X = 2^{n+1}, Y = 2^n) \text{ for } n \geq 0.$$

(a) Show that $\mathbb{P}(X = 1) = \frac{1}{4}$ and

$$\mathbb{P}(X = 2^n) = \frac{3}{4}2^{-n} \text{ for } n \geq 1.$$

(b) Show that $\mathbb{P}(Y = 2|X = 1) = 1$ and

$$\mathbb{P}(Y = 2^{n+1}|X = 2^n) = \frac{1}{3} = 1 - \mathbb{P}(Y = 2^{n-1}|X = 2^n) \text{ for } n \geq 1.$$

(c) Let $Z = Y - X$. Show that $\mathbb{E}(Z|X = 1) = 1$ and

$$\mathbb{E}(Z|X = 2^n) = 0 \text{ for } n \geq 1.$$

(d) From part (c) we have $\mathbb{E}(Z|X) = \mathbb{1}_{\{X=1\}}$ and hence

$$\mathbb{E}(Z) = \mathbb{E}[\mathbb{E}(Z|X)] = \frac{1}{4} > 0.$$

However, by symmetry we also have $\mathbb{E}(Z|Y) = -\mathbb{1}_{\{Y=1\}}$ and

$$\mathbb{E}(Z) = \mathbb{E}[\mathbb{E}(Z|Y)] = -\frac{1}{4} < 0.$$

What has gone wrong?!

Solution 11. (a)

$$\begin{aligned}\mathbb{P}(X = 2^n) &= \mathbb{P}(X = 2^n, Y = 2^{n+1}) + \mathbb{P}(X = 2^n, Y = 2^{n-1}) \\ &= \frac{1}{4}2^{-n} + \frac{1}{4}2^{-(n-1)}\mathbb{1}_{\{n\geq1\}} \\ &= \frac{1}{4}2^{-n}(1 + 2\mathbb{1}_{\{n\geq1\}})\end{aligned}$$

(b) By Bayes' formula

$$\begin{aligned}\mathbb{P}(Y = 2^{n+1}|X = 2^n) &= \frac{\mathbb{P}(X = 2^n, Y = 2^{n+1})}{\mathbb{P}(X = 2^n)} \\ &= \frac{1}{1 + 2\mathbb{1}_{\{n\geq1\}}}\end{aligned}$$

by part (a).
(c) Using the formula from part (b) yields

$$\begin{aligned}\mathbb{E}[Z|X = 2^n] &= 2^{n+1}\mathbb{P}(Y = 2^{n+1}|X = 2^n) + 2^{n-1}\mathbb{P}(Y = 2^{n-1}|X = 2^n) - 2^n \\ &= 2^n\left(\frac{1 - \mathbb{1}_{\{n\geq1\}}}{1 + 2\mathbb{1}_{\{n\geq1\}}}\right) \\ &= \mathbb{1}_{\{n=0\}}\end{aligned}$$

(d) The problem is that Z is not integrable. Indeed,

$$\begin{aligned}\mathbb{E}(Z^+) &= \sum_{n=0}^{\infty}(2^{n+1} - 2^n)\mathbb{P}(X = 2^n, Y = 2^{n+1}) \\ &= \sum_{n=0}^{\infty}\frac{1}{2} \\ &= \infty\end{aligned}$$

Notice that the conditional expectation (given an event)

$$\mathbb{E}(Z|X = 2^n) = \frac{\mathbb{E}(Z\mathbb{1}_{\{X=2^n\}})}{\mathbb{P}(X = 2^n)}$$

is defined, but the conditional expectation (given a sigma-field) $\mathbb{E}(Z|X)$ is *not* defined[1].

[1]In fact, it is possible to define a generalised notion of conditional expectation so that $\mathbb{E}(Z|X) = \mathbb{1}_{\{X=1\}}$. But this definition is fickle, and in particular, it would not obey the tower property of conditional expectation.