Paper 1, Section I

6H Statistics

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables, where $n \geqslant 3$ and $p \in (0,1)$ is unknown.

(a) What does it mean for a statistic T to be *sufficient* for p? Find such a sufficient statistic T.

(b) State and prove the Rao–Blackwell theorem.

(c) By considering the estimator X_1X_2 of p^2, find an unbiased estimator of p^2 that is a function of the statistic T found in part (a), and has variance strictly smaller than that of X_1X_2.

Paper 2, Section I

6H Statistics

The efficacy of a new drug was tested as follows. Fifty patients were given the drug, and another fifty patients were given a placebo. A week later, the numbers of patients whose symptoms had gone entirely, improved, stayed the same and got worse were recorded, as summarised in the following table.

	Drug	Placebo
symptoms gone	14	6
improved	21	19
same	10	10
worse	5	15

Conduct a 5% significance level test of the null hypothesis that the medicine and placebo have the same effect, against the alternative that their effects differ.

[*Hint: You may find some of the following values relevant:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_6^2	χ_8^2
95th percentile	3.84	5.99	7.81	9.48	12.59	15.51

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Paper 1, Section II
18H Statistics

(a) Show that if $W_1, \ldots, W_n$ are independent random variables with common Exp(1) distribution, then $\sum_{i=1}^n W_i \sim \Gamma(n,1)$. [*Hint: If* $W \sim \Gamma(\alpha, \lambda)$ *then* $\mathbb{E}e^{tW} = \{\lambda/(\lambda - t)\}^\alpha$ *if* $t < \lambda$ *and* ∞ *otherwise.*]

(b) Show that if $X \sim U(0,1)$ then $-\log X \sim \text{Exp}(1)$.

(c) State the Neyman–Pearson lemma.

(d) Let $X_1, \ldots, X_n$ be independent random variables with common density proportional to $x^\theta \mathbf{1}_{(0,1)}(x)$ for $\theta \geqslant 0$. Find a most powerful test of size α of $H_0 : \theta = 0$ against $H_1 : \theta = 1$, giving the critical region in terms of a quantile of an appropriate gamma distribution. Find a uniformly most powerful test of size α of $H_0 : \theta = 0$ against $H_1 : \theta > 0$.

Paper 3, Section II
18H Statistics

Consider the normal linear model $Y = X\beta + \varepsilon$ where X is a known $n \times p$ design matrix with $n-2 > p \geqslant 1$, $\beta \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \sim N_n(0, \sigma^2 I)$ is a vector of normal errors with each component having variance $\sigma^2 > 0$. Suppose X has full column rank.

(i) Write down the maximum likelihood estimators, $\hat{\beta}$ and $\hat{\sigma}^2$, for β and σ^2 respectively. [You need not derive these.]

(ii) Show that $\hat{\beta}$ is independent of $\hat{\sigma}^2$.

(iii) Find the distributions of $\hat{\beta}$ and $n\hat{\sigma}^2/\sigma^2$.

(iv) Consider the following test statistic for testing the null hypothesis $H_0 :\ \beta = 0$ against the alternative $\beta \neq 0$:

$$T := \frac{\|\hat{\beta}\|^2}{n\hat{\sigma}^2}.$$

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p > 0$ be the eigenvalues of $X^T X$. Show that under H_0, T has the same distribution as

$$\frac{\sum_{j=1}^p \lambda_j^{-1} W_j}{Z}$$

where $Z \sim \chi^2_{n-p}$ and $W_1, \ldots, W_p$ are independent χ^2_1 random variables, independent of Z.

[*Hint: You may use the fact that* $X = UDV^T$ *where* $U \in \mathbb{R}^{n \times p}$ *has orthonormal columns,* $V \in \mathbb{R}^{p \times p}$ *is an orthogonal matrix and* $D \in \mathbb{R}^{p \times p}$ *is a diagonal matrix with* $D_{ii} = \sqrt{\lambda_i}$.]

(v) Find $\mathbb{E}T$ when $\beta \neq 0$. [*Hint: If* $R \sim \chi^2_\nu$ *with* $\nu > 2$, *then* $\mathbb{E}(1/R) = 1/(\nu - 2)$.]

Paper 4, Section II

17H Statistics

Suppose we wish to estimate the probability $\theta \in (0,1)$ that a potentially biased coin lands heads up when tossed. After n independent tosses, we observe X heads.

(a) Write down the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) Find the mean squared error $f(\theta)$ of $\hat{\theta}$ as a function of θ. Compute $\sup_{\theta \in (0,1)} f(\theta)$.

(c) Suppose a uniform prior is placed on θ. Find the Bayes estimator of θ under squared error loss $L(\theta, a) = (\theta - a)^2$.

(d) Now find the Bayes estimator $\tilde{\theta}$ under the loss $L(\theta, a) = \theta^{\alpha-1}(1-\theta)^{\beta-1}(\theta - a)^2$, where $\alpha, \beta \geqslant 1$. Show that

$$\tilde{\theta} = w\hat{\theta} + (1-w)\theta_0, \qquad (*)$$

where w and θ_0 depend on n, α and β.

(e) Determine the mean squared error $g_{w,\theta_0}(\theta)$ of $\tilde{\theta}$ as defined by (*).

(f) For what range of values of w do we have $\sup_{\theta \in (0,1)} g_{w,1/2}(\theta) \leqslant \sup_{\theta \in (0,1)} f(\theta)$?

[*Hint: The mean of a Beta*(a,b) *distribution is* $a/(a+b)$ *and its density* $p(u)$ *at* $u \in [0,1]$ *is* $c_{a,b}u^{a-1}(1-u)^{b-1}$, *where* $c_{a,b}$ *is a normalising constant.*]