Paper 1, Section I

6H Statistics

Suppose $X_1, \ldots, X_n$ are independent with distribution $N(\mu, 1)$. Suppose a prior $\mu \sim N(\theta, \tau^{-2})$ is placed on the unknown parameter μ for some given deterministic $\theta \in \mathbb{R}$ and $\tau > 0$. Derive the posterior mean.

Find an expression for the mean squared error of this posterior mean when $\theta = 0$.

Paper 1, Section II

19H Statistics

Let $X_1, \ldots, X_n$ be i.i.d. $U[0, 2\theta]$ random variables, where $\theta > 0$ is unknown.

(a) Derive the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) What is a *sufficient statistic*? What is a *minimal sufficient statistic*? Is $\hat{\theta}$ sufficient for θ? Is it minimal sufficient? Answer the same questions for the sample mean $\tilde{\theta} := \sum_{i=1}^{n} X_i/n$. Briefly justify your answers.

[You may use any result from the course provided it is stated clearly.]

(c) Show that the mean squared errors of $\hat{\theta}$ and $\tilde{\theta}$ are respectively

$$\frac{2\theta^2}{(n+1)(n+2)} \qquad \text{and} \qquad \frac{\theta^2}{3n}.$$

(d) Show that for each $t \in \mathbb{R}$, $\lim_{n\to\infty} \mathbb{P}(n(1 - \hat{\theta}/\theta) \geqslant t) = h(t)$ for a function h you should specify. Give, with justification, an approximate $1 - \alpha$ confidence interval for θ whose expected length is

$$\left(\frac{n\theta}{n+1}\right)\left(\frac{\log(1/\alpha)}{n - \log(1/\alpha)}\right).$$

[*Hint:* $\lim_{n\to\infty}(1 - \frac{t}{n})^n = e^{-t}$ *for all* $t \in \mathbb{R}$.]

Paper 2, Section II

18H Statistics

Consider the general linear model $Y = X\beta^0 + \varepsilon$ where X is a known $n \times p$ design matrix with $p \geqslant 2$, $\beta^0 \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \in \mathbb{R}^n$ is a vector of stochastic errors with $\mathbb{E}(\varepsilon_i) = 0$, $\text{var}(\varepsilon_i) = \sigma^2 > 0$ and $\text{cov}(\varepsilon_i, \varepsilon_j) = 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Suppose X has full column rank.

(a) Write down the least squares estimate $\hat{\beta}$ of β^0 and show that it minimises the least squares objective $S(\beta) = \|Y - X\beta\|^2$ over $\beta \in \mathbb{R}^p$.

(b) Write down the variance–covariance matrix $\text{cov}(\hat{\beta})$.

(c) Let $\tilde{\beta} \in \mathbb{R}^p$ minimise $S(\beta)$ over $\beta \in \mathbb{R}^p$ subject to $\beta_p = 0$. Let Z be the $n \times (p-1)$ submatrix of X that excludes the final column. Write down $\text{cov}(\tilde{\beta})$.

(d) Let P and P_0 be $n \times n$ orthogonal projections onto the column spaces of X and Z respectively. Show that for all $u \in \mathbb{R}^n$, $u^T P u \geqslant u^T P_0 u$.

(e) Show that for all $x \in \mathbb{R}^p$,

$$\text{var}(x^T \tilde{\beta}) \leqslant \text{var}(x^T \hat{\beta}).$$

[*Hint: Argue that* $x = X^T u$ *for some* $u \in \mathbb{R}^n$.]