Paper 1, Section I

7H Statistics

Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ random variables.

(a) Compute the MLEs $\widehat{\mu}$, $\widehat{\sigma}^2$ for the unknown parameters μ, σ^2.

(b) Give the definition of an *unbiased estimator*. Determine whether $\widehat{\mu}$, $\widehat{\sigma}^2$ are unbiased estimators for μ, σ^2.

Paper 2, Section I

8H Statistics

Suppose that $X_1, \ldots, X_n$ are i.i.d. coin tosses with probability θ of obtaining a head.

(a) Compute the posterior distribution of θ given the observations $X_1, \ldots, X_n$ in the case of a uniform prior on $[0,1]$.

(b) Give the definition of the *quadratic error loss function.*

(c) Determine the value $\widehat{\theta}$ of θ which minimizes the quadratic error loss function. Justify your answer. Calculate $\mathbb{E}[\widehat{\theta}]$.

[*You may use that the* $\beta(a,b)$, $a, b > 0$, *distribution has density function on* $[0,1]$ *given by*

$$c_{a,b}\, x^{a-1}(1-x)^{b-1}$$

where $c_{a,b}$ *is a normalizing constant. You may also use without proof that the mean of a* $\beta(a,b)$ *random variable is* $a/(a+b)$.]

Paper 4, Section II

19H Statistics

Consider the linear model

$$Y_i = \beta x_i + \epsilon_i \quad \text{for} \quad i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are known and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $N(0, \sigma^2)$. We assume that the parameters β and σ^2 are unknown.

(a) Find the MLE $\widehat{\beta}$ of β. Explain why $\widehat{\beta}$ is the same as the least squares estimator of β.

(b) State and prove the Gauss–Markov theorem for this model.

(c) For each value of $\theta \in \mathbb{R}$ with $\theta \neq 0$, determine the unbiased linear estimator $\widetilde{\beta}$ of β which minimizes

$$\mathbb{E}_{\beta,\sigma^2}[\exp(\theta(\widetilde{\beta} - \beta))]\,.$$

Paper 1, Section II

19H Statistics

State and prove the Neyman–Pearson lemma.

Suppose that $X_1, \ldots, X_n$ are i.i.d. $\exp(\lambda)$ random variables where λ is an unknown parameter. We wish to test the hypothesis $H_0 : \lambda = \lambda_0$ against the hypothesis $H_1 : \lambda = \lambda_1$ where $\lambda_1 < \lambda_0$.

(a) Find the critical region of the likelihood ratio test of size α in terms of the sample mean $\overline{X}$.

(b) Define the *power function* of a hypothesis test and identify the power function in the setting described above in terms of the $\Gamma(n, \lambda)$ probability distribution function. [You may use without proof that $X_1 + \cdots + X_n$ is distributed as a $\Gamma(n, \lambda)$ random variable.]

(c) Define what it means for a hypothesis test to be *uniformly most powerful.* Determine whether the likelihood ratio test considered above is uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ against $\widetilde{H}_1 : \lambda < \lambda_0$.

Paper 3, Section II

20H Statistics

Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$. Let

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad S_{XX} = \sum_{i=1}^{n} (X_i - \overline{X})^2 .$$

(a) Compute the distributions of $\overline{X}$ and S_{XX} and show that $\overline{X}$ and S_{XX} are independent.

(b) Write down the distribution of $\sqrt{n}(\overline{X} - \mu)/\sqrt{S_{XX}/(n-1)}$.

(c) For $\alpha \in (0,1)$, find a $100(1-\alpha)\%$ confidence interval in each of the following situations:

(i) for μ when σ^2 is known;

(ii) for μ when σ^2 is not known;

(iii) for σ^2 when μ is not known.

(d) Suppose that $\widetilde{X}_1, \ldots, \widetilde{X}_{\widetilde{n}}$ are i.i.d. $N(\widetilde{\mu}, \widetilde{\sigma}^2)$. Explain how you would use the F-test to test the hypothesis $H_1 : \sigma^2 > \widetilde{\sigma}^2$ against the hypothesis $H_0 : \sigma^2 = \widetilde{\sigma}^2$. Does the F-test depend on whether μ, $\widetilde{\mu}$ are known?