Paper 1, Section I

7H Statistics

(a) State and prove the Rao–Blackwell theorem.

(b) Let $X_1, \ldots, X_n$ be an independent sample from $Poisson(\lambda)$ with $\theta = e^{-\lambda}$ to be estimated. Show that $Y = 1_{\{0\}}(X_1)$ is an unbiased estimator of θ and that $T = \sum_i X_i$ is a sufficient statistic.

What is $\mathbb{E}[Y \mid T]$?

Paper 2, Section I

8H Statistics

(a) Define a $100\gamma\%$ *confidence interval* for an unknown parameter θ.

(b) Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution $N(\mu, 1)$ with μ unknown. Find a 95% confidence interval for μ.

[You may use the fact that $\Phi(1.96) \simeq 0.975$.]

(c) Let U_1, U_2 be independent $U[\theta - 1, \theta + 1]$ with θ to be estimated. Find a 50% confidence interval for θ.

Suppose that we have two observations $u_1 = 10$ and $u_2 = 11.5$. What might be a better interval to report in this case?

Paper 4, Section II

19H Statistics

(a) State and prove the Neyman–Pearson lemma.

(b) Let X be a real random variable with density $f(x) = (2\theta x + 1 - \theta)1_{[0,1]}(x)$ with $-1 \leqslant \theta \leqslant 1$.

Find a most powerful test of size α of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

Find a uniformly most powerful test of size α of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$.

Paper 1, Section II

19H Statistics

(a) Give the definitions of a *sufficient* and a *minimal sufficient* statistic T for an unknown parameter θ.

Let $X_1, X_2, \ldots, X_n$ be an independent sample from the geometric distribution with success probability $1/\theta$ and mean $\theta > 1$, i.e. with probability mass function

$$p(m) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^{m-1} \quad \text{for } m = 1, 2, \ldots.$$

Find a minimal sufficient statistic for θ. Is your statistic a biased estimator of θ?

[You may use results from the course provided you state them clearly.]

(b) Define the *bias* of an estimator. What does it mean for an estimator to be *unbiased*?

Suppose that Y has the truncated Poisson distribution with probability mass function

$$p(y) = (e^{\theta} - 1)^{-1} \cdot \frac{\theta^y}{y!} \quad \text{for } y = 1, 2, \ldots.$$

Show that the only unbiased estimator T of $1 - e^{-\theta}$ based on Y is obtained by taking $T = 0$ if Y is odd and $T = 2$ if Y is even.

Is this a useful estimator? Justify your answer.

Paper 3, Section II

20H Statistics

Consider the general linear model

$$\boldsymbol{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where X is a known $n \times p$ matrix of full rank $p < n$, $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(0, \sigma^2 I)$ with σ^2 known and $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown vector.

(a) State without proof the Gauss–Markov theorem.

Find the maximum likelihood estimator $\widehat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$. Is it unbiased?

Let $\boldsymbol{\beta}^*$ be any unbiased estimator for $\boldsymbol{\beta}$ which is linear in (Y_i). Show that

$$\operatorname{var}(\boldsymbol{t}^T \widehat{\boldsymbol{\beta}}) \leqslant \operatorname{var}(\boldsymbol{t}^T \boldsymbol{\beta}^*)$$

for all $\boldsymbol{t} \in \mathbb{R}^p$.

(b) Suppose now that $p = 1$ and that $\boldsymbol{\beta}$ and σ^2 are both unknown. Find the maximum likelihood estimator for σ^2. What is the joint distribution of $\widehat{\boldsymbol{\beta}}$ and $\widehat{\sigma}^2$ in this case? Justify your answer.