Paper 1, Section I

7H Statistics

Let $X_1, \ldots, X_n$ be independent samples from the exponential distribution with density $f(x;\lambda) = \lambda e^{-\lambda x}$ for $x > 0$, where λ is an unknown parameter. Find the critical region of the most powerful test of size α for the hypotheses $H_0 : \lambda = 1$ versus $H_1 : \lambda = 2$. Determine whether or not this test is uniformly most powerful for testing $H_0' : \lambda \leqslant 1$ versus $H_1' : \lambda > 1$.

Paper 2, Section I

8H Statistics

The efficacy of a new medicine was tested as follows. Fifty patients were given the medicine, and another fifty patients were given a placebo. A week later, the number of patients who got better, stayed the same, or got worse was recorded, as summarised in this table:

	medicine	placebo
better	28	22
same	4	16
worse	18	12

Conduct a Pearson chi-squared test of size 1% of the hypothesis that the medicine and the placebo have the same effect.

[*Hint: You may find the following values relevant:*

Distribution	χ^2_1	χ^2_2	χ^2_3	χ^2_4	χ^2_5	χ^2_6
99% percentile	*6.63*	*9.21*	*11.34*	*13.3*	*15.09*	*16.81.*

]

Paper 4, Section II
19H Statistics
Consider the linear regression model

$$Y_i = \alpha + \beta x_i + \varepsilon_i,$$

for $i = 1, \ldots, n$, where the non-zero numbers $x_1, \ldots, x_n$ are known and are such that $x_1 + \ldots + x_n = 0$, the independent random variables $\varepsilon_1, \ldots, \varepsilon_n$ have the $N(0, \sigma^2)$ distribution, and the parameters α, β and σ^2 are unknown.

(a) Let $(\hat{\alpha}, \hat{\beta})$ be the maximum likelihood estimator of (α, β). Prove that for each i, the random variables $\hat{\alpha}$, $\hat{\beta}$ and $Y_i - \hat{\alpha} - \hat{\beta} x_i$ are uncorrelated. Using standard facts about the multivariate normal distribution, prove that $\hat{\alpha}$, $\hat{\beta}$ and $\sum_{i=1}^{n}(Y_i - \hat{\alpha} - \hat{\beta} x_i)^2$ are independent.

(b) Find the critical region of the generalised likelihood ratio test of size 5% for testing $H_0 : \alpha = 0$ versus $H_1 : \alpha \neq 0$. Prove that the power function of this test is of the form $w(\alpha, \beta, \sigma^2) = g(\alpha/\sigma)$ for some function g. [You are not required to find g explicitly.]

Paper 1, Section II
19H Statistics
(a) What does it mean to say a statistic T is *sufficient* for an unknown parameter θ? State the factorisation criterion for sufficiency and prove it in the discrete case.

(b) State and prove the Rao-Blackwell theorem.

(c) Let $X_1, \ldots, X_n$ be independent samples from the uniform distribution on $[-\theta, \theta]$ for an unknown positive parameter θ. Consider the two-dimensional statistic

$$T = (\min_i X_i, \max_i X_i).$$

Prove that T is sufficient for θ. Determine, with proof, whether or not T is minimally sufficient.

Paper 3, Section II

20H Statistics

Let $X_1, \ldots, X_n$ be independent samples from the Poisson distribution with mean θ.

(a) Compute the maximum likelihood estimator of θ. Is this estimator biased?

(b) Under the assumption that n is very large, use the central limit theorem to find an approximate 95% confidence interval for θ. [You may use the notation z_α for the number such that $\mathbb{P}(Z \geqslant z_\alpha) = \alpha$ for a standard normal $Z \sim N(0,1)$.]

(c) Now suppose the parameter θ has the $\Gamma(k, \lambda)$ prior distribution. What is the posterior distribution? What is the Bayes point estimator for θ for the quadratic loss function $L(\theta, a) = (\theta - a)^2$? Let X_{n+1} be another independent sample from the same distribution. Given $X_1, \ldots, X_n$, what is the posterior probability that $X_{n+1} = 0$?
[*Hint: The density of the* $\Gamma(k, \lambda)$ *distribution is* $f(x; k, \lambda) = \lambda^k x^{k-1} e^{-\lambda x}/\Gamma(k)$, *for* $x > 0$.]