Paper 1, Section I

7H Statistics

Suppose that $X_1, \ldots, X_n$ are independent normally distributed random variables, each with mean μ and variance 1, and consider testing $H_0 : \mu = 0$ against $H_1 : \mu = 1$. Explain what is meant by the *critical region*, the *size* and the *power* of a test.

For $0 < \alpha < 1$, derive the test that is most powerful among all tests of size at most α. Obtain an expression for the power of your test in terms of the standard normal distribution function $\Phi(\cdot)$.

[Results from the course may be used without proof provided they are clearly stated.]

Paper 2, Section I

8H Statistics

Suppose that, given θ, the random variable X has $\mathbb{P}(X = k) = e^{-\theta}\theta^k/k!$, $k = 0, 1, 2, \ldots$. Suppose that the prior density of θ is $\pi(\theta) = \lambda e^{-\lambda\theta}$, $\theta > 0$, for some known λ (> 0). Derive the posterior density $\pi(\theta \mid x)$ of θ based on the observation $X = x$.

For a given loss function $L(\theta, a)$, a statistician wants to calculate the value of a that minimises the expected posterior loss

$$\int L(\theta, a)\pi(\theta \mid x)d\theta.$$

Suppose that $x = 0$. Find a in terms of λ in the following cases:

(a) $L(\theta, a) = (\theta - a)^2$;

(b) $L(\theta, a) = |\theta - a|$.

Paper 4, Section II

19H Statistics

Consider a linear model $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\mathbf{Y}$ is an $n \times 1$ vector of observations, X is a known $n \times p$ matrix, $\boldsymbol{\beta}$ is a $p \times 1$ $(p < n)$ vector of unknown parameters and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of independent normally distributed random variables each with mean zero and unknown variance σ^2. Write down the log-likelihood and show that the maximum likelihood estimators $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ of $\boldsymbol{\beta}$ and σ^2 respectively satisfy

$$X^T X\hat{\boldsymbol{\beta}} = X^T\mathbf{Y}, \qquad \frac{1}{\hat{\sigma}^4}(\mathbf{Y} - X\hat{\boldsymbol{\beta}})^T(\mathbf{Y} - X\hat{\boldsymbol{\beta}}) = \frac{n}{\hat{\sigma}^2}$$

(^{T} denotes the transpose). Assuming that $X^T X$ is invertible, find the solutions $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ of these equations and write down their distributions.

Prove that $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ are independent.

Consider the model $Y_{ij} = \mu_i + \gamma x_{ij} + \varepsilon_{ij}$, $i = 1, 2, 3$ and $j = 1, 2, 3$. Suppose that, for all i, $x_{i1} = -1$, $x_{i2} = 0$ and $x_{i3} = 1$, and that ε_{ij}, $i, j = 1, 2, 3$, are independent $N(0, \sigma^2)$ random variables where σ^2 is unknown. Show how this model may be written as a linear model and write down $\mathbf{Y}$, X, $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$. Find the maximum likelihood estimators of μ_i $(i = 1, 2, 3)$, γ and σ^2 in terms of the Y_{ij}. Derive a $100(1-\alpha)\%$ confidence interval for σ^2 and for $\mu_2 - \mu_1$.

[You may assume that, if $\mathbf{W} = (\mathbf{W_1}^T, \mathbf{W_2}^T)^T$ is multivariate normal with $\operatorname{cov}(\mathbf{W_1}, \mathbf{W_2}) = 0$, then $\mathbf{W_1}$ and $\mathbf{W_2}$ are independent.]

Paper 1, Section II
19H Statistics

Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables each with probability mass function $\mathbb{P}(X_i = x_i) = p(x_i; \theta)$, where θ is an unknown parameter. State what is meant by a *sufficient statistic* for θ. State the factorisation criterion for a sufficient statistic. State and prove the Rao–Blackwell theorem.

Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables with

$$\mathbb{P}(X_i = x_i) = \binom{m}{x_i} \theta^{x_i}(1-\theta)^{m-x_i}, \quad x_i = 0, \ldots, m,$$

where m is a known positive integer and θ is unknown. Show that $\tilde{\theta} = X_1/m$ is unbiased for θ.

Show that $T = \sum_{i=1}^n X_i$ is sufficient for θ and use the Rao–Blackwell theorem to find another unbiased estimator $\hat{\theta}$ for θ, giving details of your derivation. Calculate the variance of $\hat{\theta}$ and compare it to the variance of $\tilde{\theta}$.

A statistician cannot remember the exact statement of the Rao–Blackwell theorem and calculates $\mathbb{E}(T \mid X_1)$ in an attempt to find an estimator of θ. Comment on the suitability or otherwise of this approach, giving your reasons.

[*Hint: If a and b are positive integers then, for $r = 0, 1, \ldots, a+b$, $\binom{a+b}{r} = \sum_{j=0}^{r} \binom{a}{j}\binom{b}{r-j}$.*]

Paper 3, Section II
20H Statistics

(a) Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables, each with density $f(x) = \theta \exp(-\theta x)$, $x > 0$ for some unknown $\theta > 0$. Use the generalised likelihood ratio to obtain a size α test of $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$.

(b) A die is loaded so that, if p_i is the probability of face i, then $p_1 = p_2 = \theta_1$, $p_3 = p_4 = \theta_2$ and $p_5 = p_6 = \theta_3$. The die is thrown n times and face i is observed x_i times. Write down the likelihood function for $\theta = (\theta_1, \theta_2, \theta_3)$ and find the maximum likelihood estimate of θ.

Consider testing whether or not $\theta_1 = \theta_2 = \theta_3$ for this die. Find the generalised likelihood ratio statistic Λ and show that

$$2 \log_e \Lambda \approx T, \quad \text{where } T = \sum_{i=1}^{3} \frac{(o_i - e_i)^2}{e_i},$$

where you should specify o_i and e_i in terms of $x_1, \ldots, x_6$. Explain how to obtain an approximate size 0.05 test using the value of T. Explain what you would conclude (and why) if $T = 2.03$.