Paper 1, Section I

7H Statistics

Let $x_1, \ldots, x_n$ be independent and identically distributed observations from a distribution with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-\mu)}, & x \geqslant \mu, \\ 0, & x < \mu, \end{cases}$$

where λ and μ are unknown positive parameters. Let $\beta = \mu + 1/\lambda$. Find the maximum likelihood estimators $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\beta}$.

Determine for each of $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\beta}$ whether or not it has a positive bias.

Paper 2, Section I

8H Statistics

State and prove the Rao–Blackwell theorem.

Individuals in a population are independently of three types $\{0, 1, 2\}$, with unknown probabilities p_0, p_1, p_2 where $p_0 + p_1 + p_2 = 1$. In a random sample of n people the ith person is found to be of type $x_i \in \{0, 1, 2\}$.

Show that an unbiased estimator of $\theta = p_0 p_1 p_2$ is

$$\hat{\theta} = \begin{cases} 1, & \text{if } (x_1, x_2, x_3) = (0, 1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that n_i of the individuals are of type i. Find an unbiased estimator of θ, say θ^*, such that $\text{var}(\theta^*) < \theta(1 - \theta)$.

Paper 4, Section II

19H Statistics

Explain the notion of a sufficient statistic.

Suppose X is a random variable with distribution F taking values in $\{1, \ldots, 6\}$, with $P(X = i) = p_i$. Let $x_1, \ldots, x_n$ be a sample from F. Suppose n_i is the number of these x_j that are equal to i. Use a factorization criterion to explain why $(n_1, \ldots, n_6)$ is sufficient for $\theta = (p_1, \ldots, p_6)$.

Let H_0 be the hypothesis that $p_i = 1/6$ for all i. Derive the statistic of the generalized likelihood ratio test of H_0 against the alternative that this is not a good fit.

Assuming that $n_i \approx n/6$ when H_0 is true and n is large, show that this test can be approximated by a chi-squared test using a test statistic

$$T = -n + \frac{6}{n}\sum_{i=1}^{6} n_i^2 .$$

Suppose $n = 100$ and $T = 8.12$. Would you reject H_0? Explain your answer.

Paper 1, Section II

19H Statistics

Consider the general linear model $Y = X\theta + \epsilon$ where X is a known $n \times p$ matrix, θ is an unknown $p \times 1$ vector of parameters, and ϵ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variance σ^2. Assume the $p \times p$ matrix $X^T X$ is invertible. Let

$$\hat{\theta} = (X^T X)^{-1} X^T Y$$
$$\hat{\epsilon} = Y - X\hat{\theta}.$$

What are the distributions of $\hat{\theta}$ and $\hat{\epsilon}$? Show that $\hat{\theta}$ and $\hat{\epsilon}$ are uncorrelated.

Four apple trees stand in a 2×2 rectangular grid. The annual yield of the tree at coordinate (i, j) conforms to the model

$$y_{ij} = \alpha_i + \beta x_{ij} + \epsilon_{ij}, \quad i, j \in \{1, 2\},$$

where x_{ij} is the amount of fertilizer applied to tree (i, j), α_1, α_2 may differ because of varying soil across rows, and the ϵ_{ij} are $N(0, \sigma^2)$ random variables that are independent of one another and from year to year. The following two possible experiments are to be compared:

$$\text{I}: \ (x_{ij}) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \text{II}: \ (x_{ij}) = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}.$$

Represent these as general linear models, with $\theta = (\alpha_1, \alpha_2, \beta)$. Compare the variances of estimates of β under I and II.

With II the following yields are observed:

$$(y_{ij}) = \begin{pmatrix} 100 & 300 \\ 600 & 400 \end{pmatrix}.$$

Forecast the total yield that will be obtained next year if no fertilizer is used. What is the 95% predictive interval for this yield?

Paper 3, Section II

20H Statistics

Suppose x_1 is a single observation from a distribution with density f over $[0,1]$. It is desired to test $H_0 : f(x) = 1$ against $H_1 : f(x) = 2x$.

Let $\delta : [0,1] \to \{0,1\}$ define a test by $\delta(x_1) = i \iff$ 'accept H_i'. Let $\alpha_i(\delta) = P(\delta(x_1) = 1 - i \mid H_i)$. State the Neyman-Pearson lemma using this notation.

Let δ be the best test of size 0.10. Find δ and $\alpha_1(\delta)$.

Consider now $\delta : [0,1] \to \{0,1,\star\}$ where $\delta(x_1) = \star$ means 'declare the test to be inconclusive'. Let $\gamma_i(\delta) = P(\delta(x) = \star \mid H_i)$. Given prior probabilities π_0 for H_0 and $\pi_1 = 1 - \pi_0$ for H_1, and some w_0, w_1, let

$$\text{cost}(\delta) = \pi_0\big(w_0\alpha_0(\delta) + \gamma_0(\delta)\big) + \pi_1\big(w_1\alpha_1(\delta) + \gamma_1(\delta)\big).$$

Let $\delta^*(x_1) = i \iff x_1 \in A_i$, where $A_0 = [0, 0.5)$, $A_\star = [0.5, 0.6)$, $A_1 = [0.6, 1]$. Prove that for each value of $\pi_0 \in (0,1)$ there exist w_0, w_1 (depending on π_0) such that $\text{cost}(\delta^*) = \min_\delta \text{cost}(\delta)$. [*Hint*: $w_0 = 1 + 2(0.6)(\pi_1/\pi_0)$.]

Hence prove that if δ is any test for which

$$\alpha_i(\delta) \leqslant \alpha_i(\delta^*), \quad i = 0, 1$$

then $\gamma_0(\delta) \geqslant \gamma_0(\delta^*)$ and $\gamma_1(\delta) \geqslant \gamma_1(\delta^*)$.