Paper 1, Section I

7H Statistics

Consider the experiment of tossing a coin n times. Assume that the tosses are independent and the coin is biased, with unknown probability p of heads and $1-p$ of tails. A total of X heads is observed.

(i) What is the maximum likelihood estimator $\widehat{p}$ of p?

Now suppose that a Bayesian statistician has the Beta(M, N) prior distribution for p.

(ii) What is the posterior distribution for p?

(iii) Assuming the loss function is $L(p, a) = (p-a)^2$, show that the statistician's point estimate for p is given by

$$\frac{M+X}{M+N+n}.$$

[The Beta(M, N) distribution has density $\dfrac{\Gamma(M+N)}{\Gamma(M)\Gamma(N)} x^{M-1}(1-x)^{N-1}$ for $0 < x < 1$ and mean $\dfrac{M}{M+N}$.]

Paper 2, Section I

8H Statistics

Let $X_1, \ldots, X_n$ be random variables with joint density function $f(x_1, \ldots, x_n; \theta)$, where θ is an unknown parameter. The null hypothesis $H_0 : \theta = \theta_0$ is to be tested against the alternative hypothesis $H_1 : \theta = \theta_1$.

(i) Define the following terms: critical region, Type I error, Type II error, size, power.

(ii) State and prove the Neyman–Pearson lemma.

Paper 1, Section II

19H Statistics

Let $X_1, \ldots, X_n$ be independent random variables with probability mass function $f(x;\theta)$, where θ is an unknown parameter.

(i) What does it mean to say that T is a sufficient statistic for θ? State, but do not prove, the factorisation criterion for sufficiency.

(ii) State and prove the Rao–Blackwell theorem.

Now consider the case where $f(x;\theta) = \dfrac{1}{x!}(-\log\theta)^x\theta$ for non-negative integer x and $0 < \theta < 1$.

(iii) Find a one-dimensional sufficient statistic T for θ.

(iv) Show that $\widetilde{\theta} = \mathbb{1}_{\{X_1=0\}}$ is an unbiased estimator of θ.

(v) Find another unbiased estimator $\widehat{\theta}$ which is a function of the sufficient statistic T and that has smaller variance than $\widetilde{\theta}$. You may use the following fact without proof: $X_1 + \cdots + X_n$ has the Poisson distribution with parameter $-n\log\theta$.

Paper 3, Section II

20H Statistics

Consider the general linear model

$$Y = X\beta + \epsilon$$

where X is a known $n \times p$ matrix, β is an unknown $p \times 1$ vector of parameters, and ϵ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variance σ^2. Assume the $p \times p$ matrix $X^T X$ is invertible.

(i) Derive the least squares estimator $\widehat{\beta}$ of β.

(ii) Derive the distribution of $\widehat{\beta}$. Is $\widehat{\beta}$ an unbiased estimator of β?

(iii) Show that $\frac{1}{\sigma^2}\|Y - X\widehat{\beta}\|^2$ has the χ^2 distribution with k degrees of freedom, where k is to be determined.

(iv) Let $\widetilde{\beta}$ be an unbiased estimator of β of the form $\widetilde{\beta} = CY$ for some $p \times n$ matrix C. By considering the matrix $\mathbb{E}[(\widehat{\beta} - \widetilde{\beta})(\widehat{\beta} - \beta)^T]$ or otherwise, show that $\widehat{\beta}$ and $\widehat{\beta} - \widetilde{\beta}$ are independent.

[You may use standard facts about the multivariate normal distribution as well as results from linear algebra, including the fact that $I - X(X^T X)^{-1} X^T$ is a projection matrix of rank $n - p$, as long as they are carefully stated.]

Paper 4, Section II

19H Statistics

Consider independent random variables $X_1, \ldots, X_n$ with the $N(\mu_X, \sigma_X^2)$ distribution and $Y_1, \ldots, Y_n$ with the $N(\mu_Y, \sigma_Y^2)$ distribution, where the means μ_X, μ_Y and variances σ_X^2, σ_Y^2 are unknown. Derive the generalised likelihood ratio test of size α of the null hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$ against the alternative $H_1 : \sigma_X^2 \neq \sigma_Y^2$. Express the critical region in terms of the statistic $T = \dfrac{S_{XX}}{S_{XX} + S_{YY}}$ and the quantiles of a beta distribution, where

$$S_{XX} = \sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2 \quad \text{and} \quad S_{YY} = \sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2 .$$

[You may use the following fact: if $U \sim \Gamma(a, \lambda)$ and $V \sim \Gamma(b, \lambda)$ are independent, then $\dfrac{U}{U+V} \sim \text{Beta}(a, b)$.]