1/I/7H **Statistics**

A Bayesian statistician observes a random sample $X_1, \ldots, X_n$ drawn from a $N(\mu, \tau^{-1})$ distribution. He has a prior density for the unknown parameters μ, τ of the form

$$\pi_0(\mu, \tau) \propto \tau^{\alpha_0 - 1} \exp\left(-\tfrac{1}{2} K_0 \tau (\mu - \mu_0)^2 - \beta_0 \tau\right) \sqrt{\tau},$$

where α_0, β_0, μ_0 and K_0 are constants which he chooses. Show that after observing $X_1, \ldots, X_n$ his posterior density $\pi_n(\mu, \tau)$ is again of the form

$$\pi_n(\mu, \tau) \propto \tau^{\alpha_n - 1} \exp\left(-\tfrac{1}{2} K_n \tau (\mu - \mu_n)^2 - \beta_n \tau\right) \sqrt{\tau},$$

where you should find explicitly the form of α_n, β_n, μ_n and K_n.

1/II/18H **Statistics**

Suppose that $X_1, \ldots, X_n$ is a sample of size n with common $N(\mu_X, 1)$ distribution, and $Y_1, \ldots, Y_n$ is an independent sample of size n from a $N(\mu_Y, 1)$ distribution.

(i) Find (with careful justification) the form of the size-α likelihood–ratio test of the null hypothesis $H_0 : \mu_Y = 0$ against alternative $H_1 : (\mu_X, \mu_Y)$ unrestricted.

(ii) Find the form of the size-α likelihood–ratio test of the hypothesis

$$H_0 : \mu_X \geqslant A, \mu_Y = 0,$$

against $H_1 : (\mu_X, \mu_Y)$ unrestricted, where A is a given constant.

Compare the critical regions you obtain in (i) and (ii) and comment briefly.

2/II/19H **Statistics**

Suppose that the joint distribution of random variables X, Y taking values in $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ is given by the joint probability generating function

$$\varphi(s,t) \equiv E\left[s^X t^Y\right] = \frac{1-\alpha-\beta}{1-\alpha s-\beta t},$$

where the unknown parameters α and β are positive, and satisfy the inequality $\alpha+\beta<1$. Find $E(X)$. Prove that the probability mass function of (X, Y) is

$$f(x, y \mid \alpha, \beta) = (1-\alpha-\beta)\binom{x+y}{x}\alpha^x\beta^y \qquad (x, y \in \mathbb{Z}^+),$$

and prove that the maximum-likelihood estimators of α and β based on a sample of size n drawn from the distribution are

$$\hat{\alpha} = \frac{\overline{X}}{1+\overline{X}+\overline{Y}}, \qquad \hat{\beta} = \frac{\overline{Y}}{1+\overline{X}+\overline{Y}},$$

where $\overline{X}$ (respectively, $\overline{Y}$) is the sample mean of $X_1, \ldots, X_n$ (respectively, $Y_1, \ldots, Y_n$).

By considering $\hat{\alpha}+\hat{\beta}$ or otherwise, prove that the maximum-likelihood estimator is biased. Stating clearly any results to which you appeal, prove that as $n \to \infty$, $\hat{\alpha} \to \alpha$, making clear the sense in which this convergence happens.

3/I/8H **Statistics**

If $X_1, \ldots, X_n$ is a sample from a density $f(\cdot|\theta)$ with θ unknown, what is a 95% confidence set for θ?

In the case where the X_i are independent $N(\mu, \sigma^2)$ random variables with σ^2 known, μ unknown, find (in terms of σ^2) how large the size n of the sample must be in order for there to exist a 95% confidence interval for μ of length no more than some given $\varepsilon > 0$.

[*Hint: If* $Z \sim N(0,1)$ *then* $P(Z > 1.960) = 0.025$.]

4/II/19H **Statistics**

(i) Consider the linear model

$$Y_i = \alpha + \beta x_i + \varepsilon_i \,,$$

where observations Y_i, $i = 1, \ldots, n$, depend on known explanatory variables x_i, $i = 1, \ldots, n$, and independent $N(0, \sigma^2)$ random variables ε_i, $i = 1, \ldots, n$.

Derive the maximum-likelihood estimators of α, β and σ^2.

Stating clearly any results you require about the distribution of the maximum-likelihood estimators of α, β and σ^2, explain how to construct a test of the hypothesis that $\alpha = 0$ against an unrestricted alternative.

(ii) A simple ballistic theory predicts that the range of a gun fired at angle of elevation θ should be given by the formula

$$Y = \frac{V^2}{g} \sin 2\theta \,,$$

where V is the muzzle velocity, and g is the gravitational acceleration. Shells are fired at 9 different elevations, and the ranges observed are as follows:

θ (degrees)	5	15	25	35	45	55	65	75	85
$\sin 2\theta$	0.1736	0.5	0.7660	0.9397	1	0.9397	0.7660	0.5	0.1736
Y (m)	4322	11898	17485	20664	21296	19491	15572	10027	3458

The model

$$Y_i = \alpha + \beta \sin 2\theta_i + \varepsilon_i \qquad (*)$$

is proposed. Using the theory of part (i) above, find expressions for the maximum-likelihood estimators of α and β.

The t-test of the null hypothesis that $\alpha = 0$ against an unrestricted alternative does not reject the null hypothesis. Would you be willing to accept the model $(*)$? Briefly explain your answer.

[*You may need the following summary statistics of the data. If* $x_i = \sin 2\theta_i$, *then* $\bar{x} \equiv n^{-1} \sum x_i = 0.63986$, $\bar{Y} = 13802$, $S_{xx} \equiv \sum (x_i - \bar{x})^2 = 0.81517$, $S_{xy} = \sum Y_i (x_i - \bar{x}) = 17186$.]