1/I/7C Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from the $N(\mu, \sigma^2)$ distribution where μ and σ^2 are unknown. Use the generalized likelihood-ratio test to derive the form of a test of the hypothesis $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

Explain carefully how the test should be implemented.

1/II/18C Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables with

$$\mathbb{P}(X_i = 1) = \theta = 1 - \mathbb{P}(X_i = 0) \ ,$$

where θ is an unknown parameter, $0 < \theta < 1$, and $n \geqslant 2$. It is desired to estimate the quantity $\phi = \theta(1-\theta) = n\mathbb{V}\mathrm{ar}\left(\left(X_1 + \cdots + X_n\right)/n\right)$.

(i) Find the maximum-likelihood estimate, $\hat{\phi}$, of ϕ.

(ii) Show that $\hat{\phi}_1 = X_1\left(1 - X_2\right)$ is an unbiased estimate of ϕ and hence, or otherwise, obtain an unbiased estimate of ϕ which has smaller variance than $\hat{\phi}_1$ and which is a function of $\hat{\phi}$.

(iii) Now suppose that a Bayesian approach is adopted and that the prior distribution for θ, $\pi(\theta)$, is taken to be the uniform distribution on $(0,1)$. Compute the Bayes point estimate of ϕ when the loss function is $L(\phi, a) = (\phi - a)^2$.

[*You may use that fact that when r, s are non-negative integers,*

$$\int_0^1 x^r (1-x)^s dx = r!s!/(r+s+1)! \quad]$$

2/II/19C Statistics

State and prove the Neyman–Pearson lemma.

Suppose that X is a random variable drawn from the probability density function

$$f(x \mid \theta) = \tfrac{1}{2} |\, x \,|^{\theta-1} e^{-|x|} / \Gamma(\theta), \quad -\infty < x < \infty \,,$$

where $\Gamma(\theta) = \int\limits_0^\infty y^{\theta-1} e^{-y} dy$ and $\theta \geqslant 1$ is unknown. Find the most powerful test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$. Express the power of the test as a function of α.

Is your test uniformly most powerful for testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$? Explain your answer carefully.

3/I/8C **Statistics**

Light bulbs are sold in packets of 3 but some of the bulbs are defective. A sample of 256 packets yields the following figures for the number of defectives in a packet:

No. of defectives	0	1	2	3
No. of packets	116	94	40	6

Test the hypothesis that each bulb has a constant (but unknown) probability θ of being defective independently of all other bulbs.

[*Hint: You may wish to use some of the following percentage points:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_4^2	t_1	t_2	t_3	t_4
90% *percentile*	2·71	4·61	6·25	7·78	3·08	1·89	1·64	1·53
95% *percentile*	3·84	5·99	7·81	9·49	6·31	2·92	2·35	2·13

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4/II/19C **Statistics**

Consider the linear regression model

$$Y_i = \alpha + \beta x_i + \epsilon_i, \qquad 1 \leqslant i \leqslant n\,,$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, $x_1, \ldots, x_n$ are known real numbers with $\sum_{i=1}^n x_i = 0$ and α, β and σ^2 are unknown.

(i) Find the least-squares estimates $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β, respectively, and explain why in this case they are the same as the maximum-likelihood estimates.

(ii) Determine the maximum-likelihood estimate $\widehat{\sigma}^2$ of σ^2 and find a multiple of it which is an unbiased estimate of σ^2.

(iii) Determine the joint distribution of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\sigma}^2$.

(iv) Explain carefully how you would test the hypothesis $H_0 : \alpha = \alpha_0$ against the alternative $H_1 : \alpha \neq \alpha_0$.