1/I/10H **Statistics**

Use the generalized likelihood-ratio test to derive Student's t-test for the equality of the means of two populations. You should explain carefully the assumptions underlying the test.

1/II/21H **Statistics**

State and prove the Rao–Blackwell Theorem.

Suppose that $X_1, X_2, \ldots, X_n$ are independent, identically-distributed random variables with distribution

$$P(X_1 = r) = p^{r-1}(1-p), \quad r = 1, 2, \ldots,$$

where p, $0 < p < 1$, is an unknown parameter. Determine a one-dimensional sufficient statistic, T, for p.

By first finding a simple unbiased estimate for p, or otherwise, determine an unbiased estimate for p which is a function of T.

2/I/10H **Statistics**

A study of 60 men and 90 women classified each individual according to eye colour to produce the figures below.

	Blue	Brown	Green
Men	20	20	20
Women	20	50	20

Explain how you would analyse these results. You should indicate carefully any underlying assumptions that you are making.

A further study took 150 individuals and classified them both by eye colour and by whether they were left or right handed to produce the following table.

	Blue	Brown	Green
Left Handed	20	20	20
Right Handed	20	50	20

How would your analysis change? You should again set out your underlying assumptions carefully.

[*You may wish to note the following percentiles of the* χ^2 *distribution.*

	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
95% percentile	3.84	5.99	7.81	9.49	11.07	12.59
99% percentile	6.64	9.21	11.34	13.28	15.09	16.81

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2/II/21H **Statistics**

Defining carefully the terminology that you use, state and prove the Neyman–Pearson Lemma.

Let X be a single observation from the distribution with density function

$$f(x \mid \theta) = \tfrac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty,$$

for an unknown real parameter θ. Find the best test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

When $\alpha = 0.05$, for which values of θ_0 and θ_1 will the power of the best test be at least 0.95?

4/I/9H **Statistics**

Suppose that $Y_1, \ldots, Y_n$ are independent random variables, with Y_i having the normal distribution with mean βx_i and variance σ^2; here β, σ^2 are unknown and $x_1, \ldots, x_n$ are known constants.

Derive the least-squares estimate of β.

Explain carefully how to test the hypothesis $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$.

4/II/19H **Statistics**

It is required to estimate the unknown parameter θ after observing X, a single random variable with probability density function $f(x \mid \theta)$; the parameter θ has the prior distribution with density $\pi(\theta)$ and the loss function is $L(\theta, a)$. Show that the optimal Bayesian point estimate minimizes the posterior expected loss.

Suppose now that $f(x \mid \theta) = \theta e^{-\theta x}$, $x > 0$ and $\pi(\theta) = \mu e^{-\mu\theta}$, $\theta > 0$, where $\mu > 0$ is known. Determine the posterior distribution of θ given X.

Determine the optimal Bayesian point estimate of θ in the cases when

(i) $L(\theta, a) = (\theta - a)^2$, and

(ii) $L(\theta, a) = |(\theta - a)/\theta|$.