1/I/3D **Statistics**

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a two-dimensional sufficient statistic for μ, quoting carefully, without proof, any result you use.

What is the maximum likelihood estimator of μ?

1/II/12D **Statistics**

What is a *simple hypothesis*? Define the terms *size* and *power* for a test of one simple hypothesis against another.

State, without proof, the Neyman–Pearson lemma.

Let X be a **single** random variable, with distribution F. Consider testing the null hypothesis $H_0 : F$ is standard normal, $N(0,1)$, against the alternative hypothesis $H_1 : F$ is double exponential, with density $\frac{1}{4}e^{-|x|/2}, x \in \mathbb{R}$.

Find the test of size $\alpha, \alpha < \frac{1}{4}$, which maximises power, and show that the power is $e^{-t/2}$, where $\Phi(t) = 1 - \alpha/2$ and Φ is the distribution function of $N(0,1)$.

[*Hint: if* $X \sim N(0,1)$, $P(|X| > 1) = 0.3174$.]

2/I/3D **Statistics**

Suppose the **single** random variable X has a uniform distribution on the interval $[0, \theta]$ and it is required to estimate θ with the loss function

$$L(\theta, a) = c(\theta - a)^2,$$

where $c > 0$.

Find the posterior distribution for θ and the optimal Bayes point estimate with respect to the prior distribution with density $p(\theta) = \theta e^{-\theta}, \ \theta > 0$.

2/II/12D Statistics

What is meant by a *generalized likelihood ratio test*? Explain in detail how to perform such a test.

Let $X_1, \ldots, X_n$ be independent random variables, and let X_i have a Poisson distribution with unknown mean $\lambda_i,\ i = 1, \ldots, n$.

Find the form of the generalized likelihood ratio statistic for testing $H_0 : \lambda_1 = \ldots = \lambda_n$, and show that it may be approximated by

$$\frac{1}{\bar{X}} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

If, for $n = 7$, you found that the value of this statistic was 27.3, would you accept H_0? Justify your answer.

4/I/3D Statistics

Consider the linear regression model

$$Y_i = \beta x_i + \epsilon_i,$$

$i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are given constants, and $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, with σ^2 unknown.

Find the least squares estimator $\widehat{\beta}$ of β. State, without proof, the distribution of $\widehat{\beta}$ and describe how you would test $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$, where β_0 is given.

4/II/12D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, where μ and σ^2 are unknown.

Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2, based on $X_1, \ldots, X_n$. Show that $\widehat{\mu}$ and $\widehat{\sigma}^2$ are independent, and derive their distributions.

Suppose now it is intended to construct a "prediction interval" $I(X_1, \ldots, X_n)$ for a future, independent, $N(\mu, \sigma^2)$ random variable X_0. We require

$$P\left\{ X_0 \in I(X_1, \ldots, X_n) \right\} = 1 - \alpha,$$

with the probability over the *joint* distribution of $X_0, X_1, \ldots, X_n$.

Let

$$I_\gamma(X_1, \ldots, X_n) = \left(\widehat{\mu} - \gamma \widehat{\sigma} \sqrt{1 + \frac{1}{n}},\ \widehat{\mu} + \gamma \widehat{\sigma} \sqrt{1 + \frac{1}{n}} \right).$$

By considering the distribution of $(X_0 - \widehat{\mu}) / (\widehat{\sigma} \sqrt{\frac{n+1}{n-1}})$, find the value of γ for which $P\{X_0 \in I_\gamma(X_1, \ldots, X_n)\} = 1 - \alpha$.