

## PRINCIPLES OF STATISTICS – EXAMPLES 3/4

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Questions by courtesy of Richard Nickl

Throughout, for observations  $X$  arising from a parametric model  $\{f(\cdot, \theta) : \theta \in \Theta\}$ ,  $\Theta \subseteq \mathbb{R}$ , the quadratic risk of a decision rule  $\delta(X)$  is defined to be  $R(\delta, \theta) = E_\theta(\delta(X) - \theta)^2$ .

1. Consider  $X|\theta \sim \text{Poisson}(\theta)$ , where  $\theta \in \Theta = [0, \infty)$ , and suppose the prior for  $\theta$  is a Gamma distribution with parameters  $(\alpha, \lambda)$ . Show that the posterior distribution  $\theta|X$  is also a Gamma distribution and find its parameters.

2. For  $n \in \mathbb{N}$  fixed, suppose  $X$  is binomially  $\text{Bin}(n, \theta)$ -distributed, where  $\theta \in \Theta = [0, 1]$ .

(a) Consider a prior for  $\theta$  from a  $\text{Beta}(a, b)$  distribution, where  $a, b > 0$ . Show that the posterior distribution is  $\text{Beta}(a+X, b+n-X)$ , and compute the posterior mean  $\bar{\theta}_n(X) = E(\theta|X)$ .

(b) Show that the maximum likelihood estimator for  $\theta$  is *not* identical to the posterior mean with “ignorant” uniform prior  $\theta \sim U[0, 1]$ .

(c) Assuming that  $X$  is sampled from a fixed  $\text{Bin}(n, \theta_0)$  distribution with  $\theta_0 \in (0, 1)$ , derive the asymptotic distribution of  $\sqrt{n}(\bar{\theta}_n(X) - \theta_0)$  as  $n \rightarrow \infty$ .

3. Let  $X_1, \dots, X_n$  be i.i.d. copies of a random variable  $X$ , and consider the Bayesian model  $X|\theta \sim N(\theta, 1)$  with prior  $\pi$  as  $\theta \sim N(\mu, v^2)$ . For  $0 < \alpha < 1$ , consider the credible interval

$$C_n = \{\theta \in \mathbb{R} : |\theta - E^\pi(\theta|X_1, \dots, X_n)| \leq R_n\},$$

where  $R_n$  is chosen such that  $\pi(C_n|X_1, \dots, X_n) = 1 - \alpha$ . Now assume  $X \sim N(\theta_0, 1)$  for some fixed  $\theta_0 \in \mathbb{R}$ , and show that, as  $n \rightarrow \infty$ ,  $P_{\theta_0}^N(\theta_0 \in C_n) \rightarrow 1 - \alpha$ .

4. In a general decision problem, show that (a) a decision rule  $\delta$  that has constant risk and is admissible is also minimax, and (b) any unique Bayes rule is admissible.

5. Consider an observation  $X$  from a parametric model  $\{f(\cdot, \theta) : \theta \in \Theta\}$  with prior  $\pi$  on  $\Theta \subseteq \mathbb{R}$  and a general risk function  $R(\delta, \theta) = E_\theta L(\delta(X), \theta)$ . Assume that there exists some decision rule  $\delta_0$  such that  $R(\delta_0, \theta) < \infty$  for all  $\theta \in \Theta$ .

(a) For the loss function  $L(a, \theta) = |a - \theta|$ , show that the Bayes rule associated with  $\pi$  equals any median of the posterior distribution  $\pi(\cdot|X)$ .

(b) For weight function  $w : \Theta \rightarrow [0, \infty)$  and loss function  $L(a, \theta) = w(\theta)[a - \theta]^2$ , show that the Bayes rule  $\delta_\pi$  associated with  $\pi$  is unique and equals

$$\delta_\pi(X) = \frac{E^\pi[w(\theta)\theta|X]}{E^\pi[w(\theta)|X]},$$

assuming that the expectations in the last ratio exist and are finite, and  $E^\pi[w(\theta)|X] > 0$ .

6. (a) Considering  $X_1, \dots, X_n$  i.i.d. from a  $N(\theta, 1)$ -model with  $\theta \in \Theta = \mathbb{R}$ , show that the maximum likelihood estimator is *not* a Bayes rule for estimating  $\theta$  in quadratic risk for any prior distribution  $\pi$ .

(b) Let  $X \sim \text{Bin}(n, \theta)$ , where  $\theta \in \Theta = [0, 1]$ . Find all prior distributions  $\pi$  on  $\Theta$  for which the maximum likelihood estimator is a Bayes rule for estimating  $\theta$  in quadratic risk.

7. Consider estimating  $\theta \in \Theta = [0, 1]$  in a  $\text{Bin}(n, \theta)$  model under the quadratic risk.

(a) Find the unique minimax estimator  $\hat{\theta}_n$  of  $\theta$ , and deduce that the maximum likelihood estimator  $\hat{\theta}_n$  of  $\theta$  is *not* minimax for a fixed sample size  $n \in \mathbb{N}$ . [Hint: Find first a Bayes rule with constant risk in  $\theta \in \Theta$ .]

(b) Show, however, that the maximum likelihood estimator dominates  $\tilde{\theta}_n$  in the large sample limit by proving that, as  $n \rightarrow \infty$ ,

$$\lim_n \frac{\sup_{\theta} R(\hat{\theta}_n, \theta)}{\sup_{\theta} R(\tilde{\theta}_n, \theta)} = 1$$

and

$$\lim_n \frac{R(\hat{\theta}_n, \theta)}{R(\tilde{\theta}_n, \theta)} < 1 \quad \text{for all } \theta \in [0, 1], \theta \neq \frac{1}{2}.$$

**8.** Consider  $X_1, \dots, X_n$  i.i.d. from a  $N(\theta, 1)$  model, where  $\theta \in \Theta = [0, \infty)$ . Show that the sample mean  $\bar{X}_n$  is inadmissible for quadratic risk, but that it is still minimax. What happens if  $\Theta = [a, b]$  for some  $0 < a < b < \infty$ ?

**9.** Let  $X$  be multivariate normal  $N(\theta, I)$ , where  $\theta \in \Theta = \mathbb{R}^p$ ,  $p \geq 3$ , and  $I$  is the  $p \times p$  identity matrix. Consider estimators

$$\tilde{\theta}^{(c)} = \left(1 - c \frac{p-2}{\|X\|^2}\right) X, \quad 0 < c < 2,$$

for  $\theta$ , under the risk function  $R(\delta, \theta) = E_{\theta} \|\delta(X) - \theta\|^2$ , where  $\|\cdot\|$  is the standard Euclidean norm on  $\mathbb{R}^p$ .

(a) Show that the James-Stein estimator  $\tilde{\theta}^{(1)}$  dominates all estimators  $\tilde{\theta}^{(c)}$ ,  $c \neq 1$ .

(b) Let  $\hat{\theta}$  be the maximum likelihood estimator of  $\theta$ . Show that, for any  $0 < c < 2$ ,

$$\sup_{\theta \in \Theta} R(\tilde{\theta}^{(c)}, \theta) = \sup_{\theta \in \Theta} R(\hat{\theta}, \theta).$$

**10.** Consider  $X_1, \dots, X_n$  i.i.d. from a  $N(\theta, 1)$  model with  $\theta \in \Theta = \mathbb{R}$ , and recall the Hodges' estimator

$$\tilde{\theta}_n = \bar{X}_n 1\{|\bar{X}_n| \geq n^{-1/4}\},$$

equal to the maximum likelihood estimator  $\bar{X}_n$  of  $\theta$  whenever  $|\bar{X}_n| \geq n^{-1/4}$ , and zero otherwise. Derive the asymptotic distribution of  $\sqrt{n}(\tilde{\theta}_n - \theta)$  as  $n \rightarrow \infty$  under  $P_{\theta}$  for every  $\theta \in \Theta$ , and compare it to the asymptotic distribution of  $\sqrt{n}(\bar{X}_n - \theta)$ . Now compute the asymptotic maximal risk

$$\limsup_n \sup_{\theta \in \Theta} E_{\theta} [(\sqrt{n}(T_n - \theta))^2]$$

for both  $T_n = \bar{X}_n$  and  $T_n = \tilde{\theta}_n$ .