Paper 1, Section II
29K Principles of Statistics

(a) Suppose that Θ is an open subset of $\mathbb{R}^p$, that $\Phi : \Theta \to \mathbb{R}$ is continuously differentiable at some $\theta_0 \in \Theta$, and that $\{\widehat{\theta}_n\}_{n \geqslant 1}$ is a sequence of random vectors in $\mathbb{R}^p$ satisfying $\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} Z$, where $Z \in \mathbb{R}^p$. Prove that

$$\sqrt{n}(\Phi(\widehat{\theta}_n) - \Phi(\theta_0)) \xrightarrow{d} \nabla_\theta \Phi(\theta_0)^T Z\,.$$

For the remainder of this problem, consider the $N(0, \sigma^2)$ model, where $\sigma \in (0, \infty)$.

(b) Derive the maximum likelihood estimator $\widehat{\sigma}_{\mathrm{MLE}}$ of σ based on an i.i.d. sample of size n from the model. What is the asymptotic distribution of $\sqrt{n}(\widehat{\sigma}_{\mathrm{MLE}} - \sigma)$? [*Hint: You may use, without proof, the fact that* $\mathbb{E}[Z^4] = 3$ *when* $Z \sim N(0,1)$.]

(c) What is the Fisher information $I(\sigma)$ (for the sample size $n = 1$)?

(d) Now consider the alternative parametrization of the model in terms of $\rho = \sigma^2$, where $\rho \in (0, \infty)$. What is the maximum likelihood estimator $\widehat{\rho}_{\mathrm{MLE}}$ of ρ?

Paper 2, Section II
29K Principles of Statistics

Suppose $X_1, \ldots, X_n$ are i.i.d. samples from a $N(\theta, 1)$ distribution. Consider an estimator $\widehat{\theta}_{a,b}$ of the form $a\bar{X}_n + b$, where $a, b \in \mathbb{R}$ and $\bar{X}_n$ denotes the sample mean. Throughout this question, we will consider risks computed with respect to the quadratic loss.

(a) Compute the risk of $\widehat{\theta}_{a,b}$ for estimating θ.

(b) Use the formula in part (a) to show that when $a > 1$, the estimator $\widehat{\theta}_{a,b}$ is inadmissible for estimating θ.

(c) Now use the formula in part (a) to show that when $a < 0$, the estimator $\widehat{\theta}_{a,b}$ is also inadmissible for estimating θ. [*Hint: Compare the estimator with the constant estimator* $\delta := \frac{-b}{a-1}$.]

(d) Prove that $\bar{X}_n$ is admissible for estimating θ. [*Hint: You may use, without proof, the general Cramér–Rao lower bound, and the facts that* $I(\theta) = 1$ *and* $\mathbb{E}_\theta[\delta(X)]$ *is differentiable for any estimator* δ *under the Gaussian model.*]

(e) Can any of the estimators considered in parts (b) and (c) be minimax for estimating θ?

Paper 3, Section II
28K Principles of Statistics

Suppose T_n is an estimator computed from n i.i.d. observations $X_1, \ldots, X_n$. Recall that the jackknife bias-corrected estimate of T_n is given by $\widetilde{T}_{\text{JACK}} = T_n - \widehat{B}_n$, where

$$\widehat{B}_n = (n-1)\left(\frac{1}{n}\sum_{i=1}^{n} T_{(-i)} - T_n\right).$$

(a) Suppose that as $n \to \infty$ the bias function $B_n(\theta) = \mathbb{E}_\theta[T_n] - \theta$ can be approximated as

$$B_n(\theta) = \frac{a}{n} + \frac{b}{n^2} + O\left(\frac{1}{n^3}\right),$$

for some $a, b \in \mathbb{R}$. Prove that

$$|\mathbb{E}[\widetilde{T}_{\text{JACK}}] - \theta| = O\left(\frac{1}{n^2}\right).$$

For the remainder of this problem, suppose $X_i \overset{i.i.d.}{\sim} N(\mu, 1)$.

(b) Consider the estimator $T_n = (\bar{X}_n)^2$ for $\theta = \mu^2$, where $\bar{X}_n$ denotes the sample mean. Compute the biases of T_n and $\widetilde{T}_{\text{JACK}}$.

(c) What is the asymptotic distribution of $\sqrt{n}(T_n - \mu^2)$?

(d) Show that $\sqrt{n}(\widetilde{T}_{\text{JACK}} - \mu^2)$ has the same asymptotic distribution as $\sqrt{n}(T_n - \mu^2)$. [*Hint: Define $g(t) = t^2$ and define $\bar{X}_{n-1,i}$ to be the sample mean of the observations with X_i excluded. Note that*

$$\widetilde{T}_{\text{JACK}} = T_n - \frac{n-1}{n}\sum_{i=1}^{n}\left(g(\bar{X}_{n-1,i}) - g(\bar{X}_n)\right)$$

and use the identities

$$\sum_{i=1}^{n}(\bar{X}_{n-1,i} - \bar{X}_n) = 0 \quad \textit{and} \quad \bar{X}_{n-1,i} - \bar{X}_n = \frac{1}{n-1}(\bar{X}_n - X_i).$$]

Paper 4, Section II
28K Principles of Statistics

(a) Suppose it is possible to generate samples from a Uniform$[0,1]$ distribution. Describe a method for generating samples from an exponential distribution with rate parameter 1, and prove that the method is valid.

(b) Recall that the accept/reject algorithm, which operates on two pdfs f and h satisfying $f \leqslant Mh$, proceeds as follows:

1. Generate $X \sim h$ and $U \sim$ Uniform$[0,1]$.
2. If $U \leqslant \frac{f(X)}{Mh(X)}$, take $Y = X$. Otherwise, return to Step 1.

Prove that the output Y has pdf f.

(c) Suppose the pdf f is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \geqslant 0.$$

Let h be the pdf of an exponential distribution with rate parameter 1. Explain how to apply the accept/reject algorithm in this special case. Identify an appropriate value for M.

(d) Compute the expected number of steps required to generate one sample from the pdf f in part (c) using the accept/reject algorithm.

(e) Let Y be a random variable generated according to the algorithm in (c). Now suppose we generate a random variable X using the following additional steps:

1. Generate $V \sim$ Uniform$[0,1]$.
2. If $V \leqslant \frac{1}{2}$, take $Z = Y$. Otherwise, take $Z = -Y$.

What is the distribution of Z?

(f) Suppose the final goal is to generate samples from the distribution of Z in part (e). Following the steps outlined in parts (a)–(e), could the efficiency of the algorithm be improved by choosing X to be an exponential random variable with rate parameter $\lambda \neq 1$?