Paper 1, Section II

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State and prove the Cramér–Rao inequality for a real-valued parameter θ. [Necessary regularity conditions need not be stated.]

In a general decision problem, define what it means for a decision rule to be *minimax*.

Let $X_1, \ldots, X_n$ be i.i.d. from a $N(\theta, 1)$ distribution, where $\theta \in \Theta = [0, \infty)$. Prove carefully that $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is minimax for quadratic risk on Θ.

Paper 2, Section II

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Consider $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ distribution with parameter $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times (0, \infty)$. Derive the likelihood ratio test statistic $\Lambda_n(\Theta, \Theta_0)$ for the composite hypothesis

$$H_0 : \sigma^2 = 1 \;\; vs. \;\; H_1 : \sigma^2 \neq 1,$$

where $\Theta_0 = \{(\mu, 1) : \mu \in \mathbb{R}\}$ is the parameter space constrained by H_0.

Prove carefully that

$$\Lambda_n(\Theta, \Theta_0) \to^d \chi_1^2 \quad \text{as } n \to \infty,$$

where χ_1^2 is a Chi-Square distribution with one degree of freedom.

Paper 3, Section II
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Let $\Theta = \mathbb{R}^p$, let $\mu > 0$ be a probability density function on Θ and suppose we are given a further auxiliary conditional probability density function $q(\cdot|t) > 0, t \in \Theta$, on Θ from which we can generate random draws. Consider a sequence of random variables $\{\vartheta_m : m \in \mathbb{N}\}$ generated as follows:

- For $m \in \mathbb{N}$ and given ϑ_m, generate a new draw $s_m \sim q(\cdot|\vartheta_m)$.
- Define

$$\vartheta_{m+1} = \begin{cases} s_m, & \text{with probability } \rho(\vartheta_m, s_m), \\ \vartheta_m, & \text{with probability } 1 - \rho(\vartheta_m, s_m) \end{cases}$$

where $\rho(t, s) = \min\left\{\frac{\mu(s)}{\mu(t)}\frac{q(t|s)}{q(s|t)}, 1\right\}$.

(i) Show that the Markov chain (ϑ_m) has invariant measure μ, that is, show that for all (measurable) subsets $B \subset \Theta$ and all $m \in \mathbb{N}$ we have

$$\int_\Theta \Pr(\vartheta_{m+1} \in B | \vartheta_m = t)\mu(t)dt = \int_B \mu(\theta)d\theta.$$

(ii) Now suppose that μ is the posterior probability density function arising in a statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}$ with observations x and a $N(0, I_p)$ prior distribution on θ. Derive a family $\{q(\cdot \mid t) : t \in \Theta\}$ such that in the above algorithm the acceptance probability $\rho(t, s)$ is a function of the likelihood ratio $f(x, s)/f(x, t)$, and for which the probability density function $q(\cdot \mid t)$ has covariance matrix $2\delta I_p$ for all $t \in \Theta$.

Paper 4, Section II
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Consider $X_1, \ldots, X_n$ drawn from a statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}, \Theta = \mathbb{R}^p$, with non-singular Fisher information matrix $I(\theta)$. For $\theta_0 \in \Theta, h \in \mathbb{R}^p$, define likelihood ratios

$$Z_n(h) = \log \frac{\prod_{i=1}^n f(X_i, \theta_0 + h/\sqrt{n})}{\prod_{i=1}^n f(X_i, \theta_0)}, \quad X_i \sim^{i.i.d.} f(\cdot, \theta_0).$$

Next consider the probability density functions $(p_h : h \in \mathbb{R}^p)$ of normal distributions $N(h, I(\theta_0)^{-1})$ with corresponding likelihood ratios given by

$$Z(h) = \log \frac{p_h(X)}{p_0(X)}, \quad X \sim p_0.$$

Show that for every fixed $h \in \mathbb{R}^p$, the random variables $Z_n(h)$ converge in distribution as $n \to \infty$ to $Z(h)$.

[*You may assume suitable regularity conditions of the model* $\{f(\cdot, \theta) : \theta \in \Theta\}$ *without specification, and results on uniform laws of large numbers from lectures can be used without proof.*]