Paper 3, Section II

25J Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. random variables from a $N(\theta, 1)$ distribution, $\theta \in \mathbb{R}$, and consider a Bayesian model $\theta \sim N(0, v^2)$ for the unknown parameter, where $v > 0$ is a fixed constant.

(a) Derive the posterior distribution $\Pi(\cdot \mid X_1, \ldots, X_n)$ of $\theta \mid X_1, \ldots, X_n$.

(b) Construct a credible set $C_n \subset \mathbb{R}$ such that

(i) $\Pi(C_n | X_1, \ldots, X_n) = 0.95$ for every $n \in \mathbb{N}$, and

(ii) for any $\theta_0 \in \mathbb{R}$,

$$P_{\theta_0}^{\mathbb{N}}(\theta_0 \in C_n) \to 0.95 \quad \text{as } n \to \infty,$$

where $P_\theta^{\mathbb{N}}$ denotes the distribution of the infinite sequence $X_1, X_2, \ldots$ when drawn independently from a fixed $N(\theta, 1)$ distribution.

[*You may use the central limit theorem.*]

Paper 2, Section II

26J Principles of Statistics

(a) State and prove the Cramér–Rao inequality in a parametric model $\{f(\theta) : \theta \in \Theta\}$, where $\Theta \subseteq \mathbb{R}$. [Necessary regularity conditions on the model need not be specified.]

(b) Let $X_1, \ldots, X_n$ be i.i.d. Poisson random variables with unknown parameter $EX_1 = \theta > 0$. For $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$ and $S^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ define

$$T_\alpha = \alpha \bar{X}_n + (1-\alpha) S^2, \quad 0 \leqslant \alpha \leqslant 1.$$

Show that $\mathrm{Var}_\theta(T_\alpha) \geqslant \mathrm{Var}_\theta(\bar{X}_n)$ for all values of α, θ.

Now suppose $\tilde{\theta} = \tilde{\theta}(X_1, \ldots, X_n)$ is an estimator of θ with possibly nonzero bias $B(\theta) = E_\theta \tilde{\theta} - \theta$. Suppose the function B is monotone increasing on $(0, \infty)$. Prove that the mean-squared errors satisfy

$$E_\theta(\tilde{\theta}_n - \theta)^2 \geqslant E_\theta(\bar{X}_n - \theta)^2 \quad \text{for all } \theta \in \Theta.$$

Paper 4, Section II

26J Principles of Statistics

Consider a decision problem with parameter space Θ. Define the concepts of a *Bayes decision rule* δ_π and of a *least favourable prior.*

Suppose π is a prior distribution on Θ such that the Bayes risk of the Bayes rule equals $\sup_{\theta\in\Theta} R(\delta_\pi, \theta)$, where $R(\delta, \theta)$ is the risk function associated to the decision problem. Prove that δ_π is least favourable.

Now consider a random variable X arising from the binomial distribution $Bin(n, \theta)$, where $\theta \in \Theta = [0, 1]$. Construct a least favourable prior for the squared risk $R(\delta, \theta) = E_\theta(\delta(X) - \theta)^2$. [You may use without proof the fact that the Bayes rule for quadratic risk is given by the posterior mean.]

Paper 1, Section II

27J Principles of Statistics

Derive the maximum likelihood estimator $\widehat{\theta}_n$ based on independent observations $X_1, \ldots, X_n$ that are identically distributed as $N(\theta, 1)$, where the unknown parameter θ lies in the parameter space $\Theta = \mathbb{R}$. Find the limiting distribution of $\sqrt{n}(\widehat{\theta}_n - \theta)$ as $n \to \infty$.

Now define

$$\begin{aligned} \widetilde{\theta}_n &= \widehat{\theta}_n \quad \text{whenever } |\widehat{\theta}_n| > n^{-1/4}, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and find the limiting distribution of $\sqrt{n}(\widetilde{\theta}_n - \theta)$ as $n \to \infty$.

Calculate

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} nE_\theta(T_n - \theta)^2$$

for the choices $T_n = \widehat{\theta}_n$ and $T_n = \widetilde{\theta}_n$. Based on the above findings, which estimator T_n of θ would you prefer? Explain your answer.

[*Throughout, you may use standard facts of stochastic convergence, such as the central limit theorem, provided they are clearly stated.*]