1/II/27I **Principles of Statistics**

State *Wilks' Theorem* on the asymptotic distribution of likelihood-ratio test statistics.

Suppose that $X_1, \ldots, X_n$ are independent with common $N(\mu, \sigma^2)$ distribution, where the parameters μ and σ are both unknown. Find the likelihood-ratio test statistic for testing $H_0 : \mu = 0$ against $H_1 : \mu$ unrestricted, and state its (approximate) distribution.

What is the form of the t-test of H_0 against H_1? Explain why for large n the likelihood-ratio test and the t-test are nearly the same.

2/II/27I **Principles of Statistics**

(i) Suppose that X is a multivariate normal vector with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\sigma^2 I$, where μ and σ^2 are both unknown, and I denotes the $d \times d$ identity matrix. Suppose that $\Theta_0 \subset \Theta_1$ are linear subspaces of $\mathbb{R}^d$ of dimensions d_0 and d_1, where $d_0 < d_1 < d$. Let P_i denote orthogonal projection onto Θ_i $(i = 0, 1)$. Carefully derive the joint distribution of $(|X - P_1 X|^2, |P_1 X - P_0 X|^2)$ under the hypothesis $H_0 : \mu \in \Theta_0$. How could you use this to make a test of H_0 against $H_1 : \mu \in \Theta_1$?

(ii) Suppose that I students take J exams, and that the mark X_{ij} of student i in exam j is modelled as

$$X_{ij} = m + \alpha_i + \beta_j + \varepsilon_{ij}$$

where $\sum_i \alpha_i = 0 = \sum_j \beta_j$, the ε_{ij} are independent $N(0, \sigma^2)$, and the parameters m, α, β and σ are unknown. Construct a test of $H_0 : \beta_j = 0$ for all j against $H_1 : \sum_j \beta_j = 0$.

3/II/26I **Principles of Statistics**

In the context of decision theory, explain the meaning of the following italicized terms: *loss function, decision rule*, the *risk* of a decision rule, a *Bayes rule* with respect to prior π, and an *admissible* rule. Explain how a Bayes rule with respect to a prior π can be constructed.

Suppose that $X_1, \ldots, X_n$ are independent with common $N(0, v)$ distribution, where $v > 0$ is supposed to have a prior density f_0. In a decision-theoretic approach to estimating v, we take a quadratic loss: $L(v, a) = (v - a)^2$. Write $X = (X_1, \ldots, X_n)$ and $|X| = (X_1^2 + \ldots + X_n^2)^{1/2}$.

By considering decision rules (estimators) of the form $\hat{v}(X) = \alpha |X|^2$, prove that if $\alpha \neq 1/(n+2)$ then the estimator $\hat{v}(X) = \alpha |X|^2$ is not Bayes, for any choice of prior f_0.

By considering decision rules of the form $\hat{v}(X) = \alpha |X|^2 + \beta$, prove that if $\alpha \neq 1/n$ then the estimator $\hat{v}(X) = \alpha |X|^2$ is not Bayes, for any choice of prior f_0.

[*You may use without proof the fact that, if Z has a $N(0,1)$ distribution, then $EZ^4 = 3$.*]

4/II/27I **Principles of Statistics**

A group of n hospitals is to be 'appraised'; the 'performance' θ_i of hospital i has a $N(0, 1/\tau)$ prior distribution, different hospitals being independent. The 'performance' cannot be measured directly, so an expensive firm of management consultants has been hired to arrive at each hospital's Standardised Index of Quality [SIQ], this being a number X_i for hospital i related to θ_i by the commercially-sensitive formula

$$X_i = \theta_i + \varepsilon_i,$$

where the ε_i are independent with common $N(0, 1/\tau_\varepsilon)$ distribution.

(i) Assume that τ and τ_ε are known. What is the posterior distribution of θ given X? Suppose that hospital j was the hospital with the lowest SIQ, with a value $X_j = x$; conditional on X, what is the distribution of θ_j?

(ii) Now, instead of assuming τ and τ_ε known, suppose that τ has a Gamma prior with parameters (α, β), density

$$f(t) = (\beta t)^{\alpha-1} \beta e^{-\beta t} / \Gamma(\alpha)$$

for known α and β, and that $\tau_\varepsilon = \kappa\tau$, where κ is a known constant. Find the posterior distribution of (θ, τ) given X. Comment briefly on the form of the distribution.