3002
Sample Size Theory

F. P. Treasure

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Hypotheses

A *test statistic* U is being used to distinguish between the *null* hypothesis (H_0) and the *alternate* hypothesis (H_1).

Under H_0, U has a Normal(μ_0, σ_0^2) distribution.

Under H_1, U has a Normal(μ_1, σ_1^2) distribution.

In general, μ_0, μ_1, σ_0^2 and σ_1^2 all depend on n, the total number of individuals in the sample. In particular, σ_0^2 and σ_1^2 decrease as n increases.

We suppose without losing generality that $\mu_0 < \mu_1$.

Test Characteristics

We require the (two-sided) probability of rejecting H_0 when H_0 is true to be α (the test's *size*).

We also require the probability of rejecting H_0 when H_1 is true to be greater than or equal to $1 - \beta$ (the test's *power*).

The test will be of form: reject H_0 in favour of H_1 if $U > C$, where C is the *critical value*.

The Critical Value

The size of the test determines the critical value. We require:

$$\mathcal{P}\left(U > C \mid \mathrm{H}_0\right) = \alpha/2.$$

We use H_0 to convert U to a standard Normal distribution, obtaining the equivalent requirement:

$$\mathcal{P}\left(\frac{U - \mu_0}{\sigma_0} > \frac{C - \mu_0}{\sigma_0} \mid \mathrm{H}_0\right) = \alpha/2.$$

and so (see 'Notes on Standard Normal Distribution' below)

$$1 - \Phi\left(\frac{C - \mu_0}{\sigma_0}\right) = \alpha/2$$

or

$$C = \mu_0 + \sigma_0 \Phi^{-1}(1 - \alpha/2)$$

Power

The critical value determines the power of the test. We require:

$$\mathcal{P}\left(U > C \mid \mathrm{H}_1\right) > 1 - \beta$$

which in terms of a standard Normal distribution is

$$\mathcal{P}\left(\frac{U-\mu_1}{\sigma_1} > \frac{C-\mu_1}{\sigma_1} \mid \mathrm{H}_1\right) \geq 1-\beta.$$

giving

$$C \leq \mu_1 - \sigma_1\Phi^{-1}(1-\beta).$$

Sample Size

The two expressions for C can be combined, eliminating C to give:

$$\sigma_0\Phi^{-1}(1-\alpha/2) + \sigma_1\Phi^{-1}(1-\beta) \leq \mu_1 - \mu_0.$$

The left hand side of this inequality normally decreases with n much more rapidly than the right hand side. The sample size is therefore derived as the lowest integral value of n such that the inequality is satisfied.

Notes on Standard Normal Distribution

If Z has a Normal distribution with mean zero and variance unity (that is, Z has a *Standard* Normal Distribution) then the distribution function Φ is defined by

$$\Phi(z) := \mathcal{P}(Z \leq z).$$

Φ has an inverse Φ^{-1}:

$$\mathcal{P}(Z \leq \Phi^{-1}(p)) = p.$$

We state the following useful results:

$$\mathcal{P}(Z \leq z) = 1 - \Phi(z)$$

and (through symmetry):

$$\Phi^{-1}(1-p) = -\Phi^{-1}(p).$$