3002
Sample Size Theory

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Hypotheses
A test statistic $U$ is being used to distinguish between the null hypothesis ($H_0$) and the alternate hypothesis ($H_1$).

Under $H_0$, $U$ has a Normal($\mu_0, \sigma^2_0$) distribution.
Under $H_1$, $U$ has a Normal($\mu_1, \sigma^2_1$) distribution.

In general, $\mu_0$, $\mu_1$, $\sigma^2_0$ and $\sigma^2_1$ all depend on $n$, the total number of individuals in the sample. In particular, $\sigma^2_0$ and $\sigma^2_1$ decrease as $n$ increases.

We suppose without losing generality that $\mu_0 < \mu_1$.

Test Characteristics

We require the (two-sided) probability of rejecting $H_0$ when $H_0$ is true to be $\alpha$ (the test’s size).

We also require the probability of rejecting $H_0$ when $H_1$ is true to be greater than or equal to $1 - \beta$ (the test’s power).

The test will be of form: reject $H_0$ in favour of $H_1$ if $U > C$, where $C$ is the critical value.

The Critical Value

The size of the test determines the critical value. We require:

$$P(U > C \mid H_0) = \alpha/2.$$ 

We use $H_0$ to convert $U$ to a standard Normal distribution, obtaining the equivalent requirement:

$$P\left(\frac{U - \mu_0}{\sigma_0} > \frac{C - \mu_0}{\sigma_0} \mid H_0\right) = \alpha/2.$$ 

and so (see ‘Notes on Standard Normal Distribution’ below)

$$1 - \Phi\left(\frac{C - \mu_0}{\sigma_0}\right) = \alpha/2$$

or

$$C = \mu_0 + \sigma_0 \Phi^{-1}(1 - \alpha/2)$$

Power

The critical value determines the power of the test. We require:

$$P(U > C \mid H_1) > 1 - \beta$$
which in terms of a standard Normal distribution is

\[ P\left( \frac{U - \mu_1}{\sigma_1} > \frac{C - \mu_1}{\sigma_1} \mid H_1 \right) \geq 1 - \beta. \]

giving

\[ C \leq \mu_1 - \sigma_1 \Phi^{-1}(1 - \beta). \]

**Sample Size**

The two expressions for \( C \) can be combined, eliminating \( C \) to give:

\[ \sigma_0 \Phi^{-1}(1 - \alpha/2) + \sigma_1 \Phi^{-1}(1 - \beta) \leq \mu_1 - \mu_0. \]

The left hand side of this inequality normally decreases with \( n \) much more rapidly than the right hand side. The sample size is therefore derived as the lowest integral value of \( n \) such that the inequality is satisfied.

**Notes on Standard Normal Distribution**

If \( Z \) has a Normal distribution with mean zero and variance unity (that is, \( Z \) has a Standard Normal Distribution) then the distribution function \( \Phi \) is defined by

\[ \Phi(z) := P(Z \leq z). \]

\( \Phi \) has an inverse \( \Phi^{-1} \):

\[ P(Z \leq \Phi^{-1}(p)) = p. \]

We state the following useful results:

\[ P(Z \leq z) = 1 - \Phi(z) \]

and (through symmetry):

\[ \Phi^{-1}(1 - p) = -\Phi^{-1}(p). \]