SMOLUCHOWSKI'S COAGULATION EQUATION: UNIQUENESS, NON-UNIQUENESS AND A HYDRODYNAMIC LIMIT FOR THE STOCHASTIC COALESCENT

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ABSTRACT. Sufficient conditions are given for existence and uniqueness in Smoluchowski's coagulation equation, for a wide class of coagulation kernels and initial mass distributions. An example of non-uniqueness is constructed. The stochastic coalescent is shown to converge weakly to the solution of Smoluchowski's equation.

1. INTRODUCTION

Coagulation of particles, in pairs and over time, within a large system of particles, is a phenomenon widely observed and widely postulated in scientific models. Examples arise in the study of aerosols, of phase separation in liquid mixtures, in polymerization and astronomy. Typically, it is argued that the rate at which pairs of particles coagulate depends, for physical reasons, in a given way, on some positive parameter associated to each particle, such as size or mass. In the models we shall consider, it is further argued that the effects of spatial fluctuations in the size or mass density are negligible—for example, by supposing that the particles perform independent random motions on a time scale faster than the process of coagulation.

The first mathematical model of this sort of process was proposed by Smoluchowski [vS16] in 1916, see also Chandrasekhar [Cha43]. Smoluchowski argued that particles of radius r would perform independent Brownian motions of variance proportional to $1/r$, so pairs of particles of radii r_1 and r_2 would meet at a rate proportional to

$$(r_1 + r_2)(1/r_1 + 1/r_2).$$

Expressed in terms of masses, this leads to the coagulation kernel

$$K(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$$

for particles of masses x and y. Then, making some implicit assumptions about ergodic averages, Smoluchowski wrote down the following infinite system of differential equations for the evolution of densities $\mu(x)$ of particles of mass $x = 1, 2, 3, \dots$

$$\frac{d}{dt}\mu_t(x) = \frac{1}{2}\sum_{y=1}^{x-1} K(y, x-y)\mu_t(y)\mu_t(x-y) - \mu_t(x)\sum_{y=1}^{\infty} K(x, y)\mu_t(y).$$

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Here, the first sum on the right corresponds to coagulation of smaller particles to produce one of mass x, whereas the second sum corresponds to removal of particles of mass x as they in turn coagulate to produce larger particles.

More generally, in other models, such systems of equations are considered for many different coagulation kernels K, see for example [Ald]. Also, analogous integro-differential equations are considered which allow for a continuum of masses x. It is known by now that, for a suitable initial mass distribution μ_0, Smoluchowski's original equations have a unique solution. Much progress has been made in determining when existence and uniqueness holds for more general coagulation kernels, see [McL62], [McL64], [Whi80], [BC90], [Hei92] for discrete mass distributions. Nevertheless many fundamental questions remain open, even for certain coagulation kernels studied extensively in applied sciences.

In this paper we give some new positive results on the existence and uniqueness problem for Smoluchowski-type equations. In particular:

- we prove existence of solutions for continuous coagulation kernels K such that
$$K(x,y)/xy \to 0 \quad \text{as } (x,y) \to \infty$$
extending a result of Jeon [Jeo] for the discrete case;
- we prove local existence and uniqueness of solutions when $K(x,y) \le \varphi(x)\varphi(y)$ for some continuous sublinear function $\varphi : E \to (0,\infty)$, provided that the initial mass distribution μ_0 satisfies
$$\int_{(0,\infty)} \varphi(x)^2 \mu_0\,(dx) < \infty;$$
- this allows us to treat the case where $K(x,y)$ blows up as $x \to 0$ or $y \to 0$, also to prove uniqueness in some cases when the mass distribution has no second, or even first, moment;
- we can do without any local regularity conditions on K;
- we do not have to assume that the initial mass distribution is discrete, nor that it has a density with respect to Lebesgue measure.

We also construct in §3 an example of a coagulation kernel K and an initial mass distribution μ_0, such that Smoluchowski's equation has at least two distinct solutions, both of which are conservative, in the sense that
$$\int_{(0,\infty)} x\mu_t\,(dx) = \int_{(0,\infty)} x\mu_0\,(dx) < \infty$$
for all t.

Then in §4 we consider a stochastic system of coagulating particles, where particles of masses x and y coagulate at a rate proportional to $K(x,y)$. We show that, when we can establish uniqueness in Smoluchowski's equation, the particle system, suitably normalized, converges weakly to the solution of the deterministic equation. Thus we obtain a statistical derivation of Smoluchowski's equation. This goes some way towards resolving Problem 10 in Aldous' survey article [Ald].

2. Existence and uniqueness in Smoluchowski's coagulation equation

Let $E = (0, \infty)$ and let $K : E \times E \to [0, \infty)$ be a symmetric measurable function, the *coagulation kernel*. Denote by $\mathcal{M} = \mathcal{M}_E$ the space of signed Radon measures on E, that is to say, those signed measures having finite total variation on each compact subset of E. Denote by $\mathcal{M}^+$ the set of (non-negative) measures in $\mathcal{M}$. If $\mu \in \mathcal{M}^+$ satisfies, for all compact sets $B \subseteq E$

$$\int_{B\times E} K(x,y)\mu\,(dx)\mu(dy) < \infty,$$

then we define $L(\mu) \in \mathcal{M}$ by

$$\langle f, L(\mu)\rangle = \frac{1}{2}\int_{E\times E}\{f(x+y) - f(x) - f(y)\}K(x,y)\mu(dx)\mu(dy)$$

for all bounded measurable functions f of compact support.

We consider the following weak form of Smoluchowski's coagulation equation

$$\mu_t = \mu_0 + \int_0^t L(\mu_s)\,ds. \tag{2.1}$$

We admit as a *local solution* any map

$$t \mapsto \mu_t : [0,T) \mapsto \mathcal{M}^+$$

where $T \in (0, \infty]$, such that:

(i) we have

$$\int_E x1_{x\le 1}\,\mu_0(dx) < \infty;$$

(ii) for all compact sets $B \subseteq E$, the following map is measurable

$$t \mapsto \mu_t(B) : [0,T) \to [0,\infty);$$

(iii) we have, for all $t < T$ and all compact sets $B \subseteq E$

$$\int_0^t \int_{B\times E} K(x,y)\mu_s(dx)\mu_s(dy)ds < \infty;$$

(iv) for all bounded measurable functions f of compact support and also for $f(x) = x1_{x\le 1}$, for all $t < T$

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, L(\mu_s)\rangle ds. \tag{2.2}$$

In the case $T = \infty$, we have a *solution.*

The condition that, for $f(x) = x1_{x\le 1}$, we have $\langle f, \mu_0\rangle < \infty$ and that (2.2) holds is a boundary condition, expressing that no mass enters at 0. We obtain an equivalent condition on replacing f by any non-vanishing sublinear function $E \to [0,\infty)$ of bounded support, which is linear near 0. A function $f : E \to [0,\infty)$ is *sublinear* if

$$f(\lambda x) \le \lambda f(x) \quad \text{for all} \quad x \in E, \lambda \ge 1.$$

Note that such a function f is always *subadditive*:

$$f(x+y) \le f(x) + f(y) \quad \text{for all} \quad x, y \in E.$$

Hence $\langle f, L(\mu)\rangle \le 0$ for all $\mu \in \mathcal{M}^+$. Note also that, if $\varphi : E \to [0,\infty)$ is *any* sublinear function and if

$$\varphi_n(x) = \begin{cases} nx\varphi(n^{-1}), & 0 < x \le n^{-1}, \\ \varphi(x), & n^{-1} < x \le n, \\ 0, & x > n, \end{cases}$$

then $\varphi_n(x) \uparrow \varphi(x)$ for all x, and φ_n is sublinear of bounded support, linear near 0. So, for $t < T$

$$\langle \varphi_n, \mu_t\rangle - \langle \varphi_n, \mu_0\rangle = \int_0^t \langle \varphi_n, L(\mu_s)\rangle \le 0.$$

Hence, using monotone convergence on the left and Fatou's lemma on the right

$$\langle \varphi, \mu_0\rangle \ge \langle \varphi, \mu_t\rangle - \int_0^t \langle \varphi, L(\mu_s)\rangle\, ds. \tag{2.3}$$

In particular, $\langle \varphi, \mu_t\rangle$ is non-increasing in t. In particular, the *total mass density*

$$\int_E x\mu_t\,(dx)$$

is non-increasing in t; if it is finite and constant, we say that $(\mu_t)_{t<T}$ is *conservative.*

Throughout this section we make the basic assumption that

$$K(x,y) \le \varphi(x)\varphi(y) \quad \text{for all } x, y \in E \tag{2.4}$$

where $\varphi : E \to (0,\infty)$ is a continuous sublinear function. We also assume that the initial measure μ_0 satisfies

$$\langle \varphi, \mu_0\rangle < \infty. \tag{2.5}$$

We call any local solution $(\mu_t)_{t<T}$ such that

$$\int_0^t \langle \varphi^2, \mu_s\rangle\, ds < \infty \quad \text{for all } t < T$$

a *strong* local solution.

Here is a summary of what is known so far about existence, uniqueness and conservation of mass in Smoluchowski's equation. The picture is more complete for discrete mass distributions—that is when μ_0 is supported on $\mathbb{N}$. Then, provided μ_0

has a finite second moment, Ball and Carr [BC90] proved existence and mass conservation when $K(x,y) \leq x+y$, Heilman [Hei92] added uniqueness under the same hypotheses. Jeon [Jeo] has recently proved global existence when $K(x,y)/xy \to 0$ as $(x,y) \to \infty$. McLeod [McL62] long ago proved local existence when $K(x,y) \leq xy$. For general mass distributions μ_0, less is known. Dubovskii and Stewart [DS96] have shown existence and uniqueness provided μ_0 has an exponential moment and a continuous density with respect to Lebesgue measure, and provided K is continuous with $K(x,y) \leq 1+x+y$. Recently, Clark and Katsouros [CK] proved existence and uniqueness for a particular choice of kernel which blows up when x or y is small.

Theorem 2.1. *Assume conditions* (2.4) *and* (2.5). *If $(\mu_t)_{t<T}$ and $(\nu_t)_{t<T}$ are local solutions, starting from μ_0, and if $(\mu_t)_{t<T}$ is strong, then $\mu_t = \nu_t$ for all $t < T$. If $\varphi(x) \geq \varepsilon x$ for all x, for some $\varepsilon > 0$, then any strong solution is conservative. Moreover, if $\langle \varphi^2, \mu_0 \rangle < \infty$, then*

(i) *there exists a unique maximal strong solution $(\mu_t)_{t<\zeta(\mu_0)}$, with $\zeta(\mu_0) \geq \langle \varphi^2, \mu_0 \rangle^{-1}$,*
(ii) *if φ^2 is sublinear or if $K(x,y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in E$, then $\zeta(\mu_0) = \infty$.*

The proof will occupy the remainder of this section. The method is to find an approximation to Smoluchowski's equation by a system depending on K and φ only through their values on a given compact set. The idea of the approximation may be readily understood in terms of the stochastic coalescent, for which a directly analogous approximation is discussed in §4. Moreover, the finite particle interpretation explains certain crucial non-negativity statements, which are given less transparent analytical proofs below.

We remark that this theorem provides examples where uniqueness holds, even when the solution fails to be conservative, in the trivial sense that the total mass density is infinite. We have not yet found an example of a strong solution which has finite initial mass density and which fails to conserve mass. The example of §3 shows, on the other hand, that uniqueness can fail while solutions remain conservative.

Let $B \subseteq E$ be compact. We will eventually pass to the limit $B \uparrow E$. Denote by $\mathcal{M}_B$ the space of finite signed measures supported on B. Note that φ is bounded on B. We define $L^B : \mathcal{M}_B \times \mathbb{R} \to \mathcal{M}_B \times \mathbb{R}$ by the requirement

$$\begin{aligned}
&\langle (f,a), L^B(\mu,\lambda) \rangle = \\
&\frac{1}{2} \int_{E \times E} \{ f(x+y) 1_{x+y \in B} + a\varphi(x+y) 1_{x+y \notin B} - f(x) - f(y) \} K(x,y) \mu(dx) \mu(dy) \\
&\qquad\qquad\qquad\qquad + \lambda \int_E \{ a\varphi(x) - f(x) \} \varphi(x) \mu(dx)
\end{aligned}$$

for all bounded measurable functions f on E and all $a \in \mathbb{R}$. Here we used the notation $\langle (f,a), (\mu,\lambda) \rangle = \langle f, \mu \rangle + a\lambda$.

Consider the equation

$$(\mu_t, \lambda_t) = (\mu_0, \lambda_0) + \int_0^t L^B(\mu_s, \lambda_s)\, ds. \tag{2.6}$$

We admit as a *local solution* any continuous map

$$t \mapsto (\mu_t, \lambda_t) : [0, T] \to \mathcal{M}_B \times \mathbb{R}$$

where $T \in (0, \infty)$, which satisfies the equation for all $t \in [0, T]$. When $[0, T]$ is replaced by $[0, \infty)$ we get the notion of *solution*.

Proposition 2.2. *Suppose $\mu_0 \in \mathcal{M}_B$ with $\mu_0 \geq 0$ and that $\lambda_0 \in [0, \infty)$. The equation (2.6) has a unique solution $(\mu_t, \lambda_t)_{t \geq 0}$ starting from (μ_0, λ_0). Moreover $\mu_t \geq 0$ and $\lambda_t \geq 0$ for all t.*

Proof. Our basic assumption (2.4) remains valid when φ is replaced by $\varphi + 1$, so we may assume without loss that $\varphi \geq 1$. By a scaling argument we may assume, also without loss, that

$$\langle \varphi, \mu_0 \rangle + \lambda_0 \leq 1$$

which implies that

$$\|\mu_0\| + |\lambda_0| \leq 1.$$

We shall show, by a standard iterative scheme, that there is a constant $T > 0$, depending only on φ and B, and a unique local solution $(\mu_t, \lambda_t)_{t \leq T}$ starting from (μ_0, λ_0). Then we shall show, moreover, that $\mu_t \geq 0$ for all $t \in [0, T]$.

First of all, let us see that this is enough to prove the proposition. If we put $f = 0$ and $a = 1$ in (2.6), we obtain

$$\frac{d}{dt}\lambda_t = \frac{1}{2}\int_{E \times E} \varphi(x+y) 1_{x+y \notin B} K(x, y)\mu_t(dx)\mu_t(dy) + \lambda_t \int_E \varphi(x)^2 \mu_t(dx).$$

So, since $\mu_t \geq 0$, we deduce $\lambda_t \geq 0$ for all t. Next, we put $f = \varphi$ and $a = 1$ to see that

$$\frac{d}{dt}(\langle \varphi, \mu_t \rangle + \lambda_t) = \frac{1}{2}\int_{E \times E} \{\varphi(x+y) - \varphi(x) - \varphi(y)\} K(x, y)\mu_t(dx)\mu_t(dy) \leq 0.$$

Hence

$$\|\mu_T\| + |\lambda_T| \leq \langle \varphi, \mu_T \rangle + \lambda_T \leq \langle \varphi, \mu_0 \rangle + \lambda_0 \leq 1.$$

We can now start again from (μ_T, λ_T) at time T to extend the solution to $[0, 2T]$, and so on, to prove the proposition.

We use the following norm on $\mathcal{M}_B \times \mathbb{R}$:

$$\|(\mu, \lambda)\| = \|\mu\| + |\lambda|.$$

We note the following estimates: there is a constant $C < \infty$, depending only on φ and B such that, for all $\mu, \mu' \in \mathcal{M}_B$ and all $\lambda, \lambda' \in \mathbb{R}$,

$$\|L^B(\mu, \lambda)\| \leq C\|(\mu, \lambda)\|^2, \tag{2.7}$$

$$\|L^B(\mu,\lambda) - L^B(\mu',\lambda')\| \le C\|(\mu,\lambda)-(\mu',\lambda')\|\,(\|(\mu,\lambda)\| + \|(\mu',\lambda')\|). \tag{2.8}$$

We turn to the iterative scheme. Set $(\mu_t^0,\lambda_t^0) = (\mu_0,\lambda_0)$ for all t and define inductively a sequence of continuous maps

$$t \mapsto (\mu_t^n,\lambda_t^n) : [0,\infty) \to \mathcal{M}_B \times \mathbb{R}$$

by

$$(\mu_t^{n+1},\lambda_t^{n+1}) = (\mu_0,\lambda_0) + \int_0^t L^B(\mu_s^n,\lambda_s^n)\,ds.$$

Set

$$f_n(t) = \|(\mu_t^n,\lambda_t^n)\|$$

then $f_0(t) = f_n(0) = \|(\mu_0,\lambda_0)\| \le 1$ and by the estimate (2.7)

$$f_{n+1}(t) \le 1 + C\int_0^t f_n(s)^2\,ds.$$

Hence

$$f_n(t) \le (1-Ct)^{-1}, \quad t \le C^{-1}$$

for all n, so, setting $T = (2C)^{-1}$, we have

$$\|(\mu_t^n,\lambda_t^n)\| \le 2, \quad t \le T. \tag{2.9}$$

Next, set $g_0(t) = f_0(t)$ and for $n \ge 1$

$$g_n(t) = \|(\mu_t^n,\lambda_t^n) - (\mu_t^{n-1},\lambda_t^{n-1})\|.$$

By the estimates (2.8) and (2.9), there is a constant $C < \infty$, depending only on φ and B, such that

$$g_{n+1}(t) \le C\int_0^t g_n(s)\,ds, \quad t \le T.$$

Hence, by the usual arguments, (μ_t^n,λ_t^n) converges in $\mathcal{M}_B\times\mathbb{R}$, uniformly in $t \le T$, to the desired local solution, which is also unique. Moreover, for some constant $C < \infty$, depending only on φ and B, we have

$$\|(\mu_t,\lambda_t)\| \le C, \quad t \le T.$$

It remains to show that $\mu_t \ge 0$ for all t. For this we need the following result.

Proposition 2.3. *Let*

$$(t,x) \mapsto f_t(x) : [0,T]\times B \to \mathbb{R}$$

be a bounded measurable function, having a bounded partial derivative $\partial f/\partial t$. Then for all $t \le T$

$$\frac{d}{dt}\langle f_t,\mu_t\rangle = \langle \partial f/\partial t, \mu_t\rangle + \langle (f_t,0), L^B(\mu_t,\lambda_t)\rangle.$$

Proof. Fix $t \leq T$ and set $\lfloor s \rfloor_n = (n/t)^{-1} \lfloor ns/t \rfloor$ and $\lceil s \rceil_n = (n/t)^{-1} \lceil ns/t \rceil$. Then

$$\langle f_t, \mu_t \rangle = \langle f_0, \mu_0 \rangle + \int_0^t \langle \partial f/\partial s, \mu_{\lfloor s \rfloor_n} \rangle \, ds + \int_0^t \langle (f_{\lceil s \rceil_n}, 0), L^B(\mu_s, \lambda_s) \rangle \, ds$$

and the proposition follows on letting $n \to \infty$. □

For $t \leq T$, set

$$\theta_t(x) = \exp \int_0^t \left(\int_E K(x,y)\mu_s(dy) + \lambda_s \varphi(x) \right) ds$$

and define $G_t : \mathcal{M}_B \to \mathcal{M}_B$ by

$$\langle f, G_t(\mu) \rangle = \frac{1}{2} \int_{E \times E} (f\theta_t)(x+y) 1_{x+y \in B} K(x,y) \theta_t(x)^{-1} \theta_t(y)^{-1} \mu(dx)\mu(dy).$$

We note that $G_t(\mu) \geq 0$ whenever $\mu \geq 0$ and, for some $C < \infty$, depending only on φ and B, we have

$$\|G_t(\mu)\| \leq C\|\mu\|^2, \ \|G_t(\mu) - G_t(\mu')\| \leq C\|\mu - \mu'\| \left(\|\mu\| + \|\mu'\| \right).$$

Set $\tilde{\mu}_t = \theta_t \mu_t$. By Proposition 2.3, for all bounded measurable functions f, we have

$$\frac{d}{dt} \langle f, \tilde{\mu}_t \rangle = \langle f \partial\theta/\partial t, \mu_t \rangle + \langle (f\theta_t, 0), L^B(\mu_t, \lambda_t) \rangle = \langle f, G_t(\tilde{\mu}_t) \rangle.$$

Define inductively a new sequence of measures $\tilde{\mu}_t^n$ by setting $\tilde{\mu}_t^0 = \mu_0$ and, for $n \geq 0$

$$\tilde{\mu}_t^{n+1} = \mu_0 + \int_0^t G_s(\tilde{\mu}_s^n) \, ds.$$

By an argument similar to that used for the original iterative scheme, we can show, first, and possibly for a smaller value of $T > 0$, but with the same dependence, that $\|\tilde{\mu}_t^n\|$ is bounded, uniformly in n, for $t \leq T$, and then that $\|\tilde{\mu}_t^n - \tilde{\mu}_t\| \to 0$ as $n \to \infty$. Since $\tilde{\mu}_t^n \geq 0$ for all n, we deduce $\tilde{\mu}_t \geq 0$ and hence $\mu_t \geq 0$ for all $t \leq T$. This completes the proof of Proposition 2.2 □

We remark that the arguments used to prove Proposition 2.2 apply with no essential change to the case where the coagulation kernel is time-dependent provided that (2.4) holds uniformly in time. We remark also that, in the iterative scheme

$$\mu_t^0 = \mu_0, \qquad \mu_t^{n+1} \ll \mu_0 + \int_0^t (\mu_s^n + \mu_s^n * \mu_s^n) \, ds$$

for all $n \geq 0$, so by induction we have

$$\mu_t^n \ll \gamma_0 = \sum_{k=1}^{\infty} \mu_0^{*k}$$

where μ_0^{*n} denotes the n-fold convolution of μ_0. On letting $n \to \infty$, we see that, if $(\mu_t, \lambda_t)_{t \geq 0}$ is the unique solution to (2.6), then $\mu_t \ll \gamma_0$. These remarks will be used in the proof of Proposition 4.2.

We now fix $\mu_0 \in \mathcal{M}^+$ with $\mu_0 \geq 0$ and $\langle \varphi, \mu_0 \rangle < \infty$. For each compact set $B \subseteq E$, let

$$\mu_0^B = 1_B \mu_0, \quad \lambda_0^B = \int_{E \setminus B} \varphi(x) \mu_0 \, (dx)$$

and denote by $(\mu_t^B, \lambda_t^B)_{t \geq 0}$ the unique solution to (2.6), starting from (μ_0^B, λ_0^B), provided by Proposition 2.2. We shall show in Proposition 2.4 that for $B \subseteq B'$ we have

$$\mu_t^B \leq \mu_t^{B'}, \quad \langle \varphi, \mu_t^B \rangle + \lambda_t^B \geq \langle \varphi, \mu_t^{B'} \rangle + \lambda_t^{B'}.$$

We shall also show in Proposition 2.5 that, for any local solution $(\nu_t)_{t<T}$ of the coagulation equation (2.1), for all $t < T$

$$\mu_t^B \leq \nu_t, \quad \langle \varphi, \mu_t^B \rangle + \lambda_t^B \geq \langle \varphi, \nu_t \rangle.$$

We now show how these facts lead to the proof of Theorem 2.1.

Set $\mu_t = \lim_{B \uparrow E} \mu_t^B$ and $\lambda_t = \lim_{B \uparrow E} \lambda_t^B$. Note that

$$\langle \varphi, \mu_t \rangle = \lim_{B \uparrow E} \langle \varphi, \mu_t^B \rangle \leq \langle \varphi, \mu_0 \rangle < \infty.$$

So, by dominated convergence, using (2.4), for all bounded measurable functions f,

$$\int_{E \times E} f(x+y) 1_{x+y \notin B} K(x,y) \mu_t^B \, (dx) \mu_t^B (dy) \to 0$$

and we can pass to the limit in (2.6) to obtain

$$\frac{d}{dt} \langle f, \mu_t \rangle = \frac{1}{2} \int_{E \times E} \{ f(x+y) - f(x) - f(y) \} K(x,y) \mu_t \, (dx) \mu_t (dy) - \lambda_t \langle f \varphi, \mu_t \rangle.$$

For any local solution $(\nu_t)_{t<T}$, for all $t < T$,

$$\mu_t \leq \nu_t, \quad \langle \varphi, \mu_t \rangle + \lambda_t \geq \langle \varphi, \nu_t \rangle.$$

Hence, if $\lambda_t = 0$ for all $t < T$, then $(\mu_t)_{t<T}$ is a local solution and moreover is the only local solution on $[0,T)$. If $(\nu_t)_{t<T}$ is a strong local solution, then

$$\int_0^t \langle \varphi^2, \mu_s \rangle \, ds \leq \int_0^t \langle \varphi^2, \nu_s \rangle \, ds < \infty$$

for all $t < T$; this allows us to pass to the limit in (2.6) to obtain

$$\frac{d}{dt} \lambda_t = \lambda_t \langle \varphi^2, \mu_t \rangle \tag{2.10}$$

and to deduce from this equation that $\lambda_t = 0$ for all $t < T$; it follows that $(\nu_t)_{t<T}$ is the only local solution on $[0,T)$. Note that, for any local solution $(\nu_t)_{t<T}$

$$\int_E x 1_{x \leq n} \nu_t \, (dx) = \int_E x 1_{x \leq n} \nu_0 \, (dx) + \frac{1}{2} \int_0^t \int_{E \times E} \{ (x+y) 1_{x+y \leq n} - x 1_{x \leq n} - y 1_{y \leq n} \} K(x,y) \nu_s(dx) \nu_s(dy) \, ds.$$

Hence, if $(\nu_t)_{t<T}$ is strong and $\varphi(x) \geq \varepsilon x$ for all x, for some $\varepsilon > 0$, then by dominated convergence the second term on the right tends to 0 as $n \to \infty$, showing that $(\nu_t)_{t<T}$ is conservative.

Suppose now that $\langle \varphi^2, \mu_0 \rangle < \infty$ and set $T = \langle \varphi^2, \mu_0 \rangle^{-1}$. For any compact set $B \subseteq E$ we have

$$\frac{d}{dt}\langle \varphi^2, \mu_t^B \rangle \leq \frac{1}{2} \int_{E\times E} \{\varphi(x+y)^2 - \varphi(x)^2 - \varphi(y)^2\} K(x,y) \mu_t^B(dx)\mu_t^B(dy) \leq \langle \varphi^2, \mu_t^B \rangle^2$$

so, for $t < T$,

$$\langle \varphi^2, \mu_t \rangle \leq \lim_{B \uparrow E} \langle \varphi^2, \mu_t^B \rangle \leq (T-t)^{-1}.$$

Hence (2.10) holds and forces $\lambda_t = 0$ for $t < T$ as above, so $(\mu_t)_{t<T}$ is a strong local solution.

If φ^2 is sublinear, then

$$\langle \varphi^2, \mu_t \rangle \leq \langle \varphi^2, \mu_0 \rangle < \infty.$$

If, on the other hand, $K(x,y) \leq \varphi(x) + \varphi(y)$, then

$$\begin{aligned} \frac{d}{dt}\langle \varphi^2, \mu_t^B \rangle &\leq \int_{E\times E} \varphi(x)\varphi(y)(\varphi(x)+\varphi(y)) \mu_t^B(dx)\mu_t^B(dy) \\ &\leq 2\langle \varphi, \mu_t^B \rangle \langle \varphi^2, \mu_t^B \rangle \leq 2\langle \varphi, \mu_0 \rangle \langle \varphi^2, \mu_t^B \rangle, \end{aligned}$$

so

$$\langle \varphi^2, \mu_t \rangle \leq \exp\{2\langle \varphi, \mu_0 \rangle t\}.$$

In either case we can deduce that $(\mu_t)_{t\geq 0}$ is a strong solution.

Proposition 2.4. *Suppose $B \subseteq B'$. Then, for all $t \geq 0$,*

$$\mu_t^B \leq \mu_t^{B'}, \quad \langle \varphi, \mu_t^B \rangle + \lambda_t^B \geq \langle \varphi, \mu_t^{B'} \rangle + \lambda_t^{B'}.$$

Proof. Set

$$\begin{aligned} \theta_t(x) &= \exp \int_0^t \left(\int_E K(x,y)\mu_s^B(dy) + \lambda_s^B \varphi(x) \right) ds, \\ \pi_t &= \theta_t(\mu_t^{B'} - \mu_t^B), \\ \chi_t &= \langle \varphi, \mu_t^B \rangle + \lambda_t^B - \langle \varphi, \mu_t^{B'} \rangle - \lambda_t^{B'}. \end{aligned}$$

Note that $\pi_0 \geq 0$ and $\chi_0 = 0$. By Proposition 2.3, for any bounded measurable function f

$$
\begin{aligned}
\frac{d}{dt}\langle f, \pi_t\rangle &= \langle f\, \partial\theta/\partial t, \mu_t^{B'} - \mu_t^B\rangle + \langle (f\theta_t, 0), L^{B'}(\mu_t^{B'}, \lambda_t^{B'}) + L^B(\mu_t^B, \lambda_t^B)\rangle \\
&= \frac{1}{2}\int_{E\times E} f\theta_t(x+y)K(x,y) \\
&\qquad\qquad \times (1_{x+y\in B'}\mu_t^{B'}(dx)\mu_t^{B'}(dy) - 1_{x+y\in B}\mu_t^B(dx)\mu_t^B(dy)) \\
&\quad + \int_{E\times E} f\theta_t(x)(\varphi(x)\varphi(y) - K(x,y))\mu_t^{B'}(dx)(\mu_t^{B'}(dy) - \mu_t^B(dy)) \\
&\quad + \chi_t \int_E f\theta_t(x)\varphi(x)\mu_t^{B'}(dx).
\end{aligned}
$$

Also

$$
\frac{d}{dt}\chi_t = \frac{1}{2}\int_{E\times E}\{\varphi(x)+\varphi(y)-\varphi(x+y)\}K(x,y)(\mu_t^{B'}(dx)\mu_t^{B'}(dy) - \mu_t^B(dx)\mu_t^B(dy)).
$$

So (π_t, χ_t) satisfies an equation of the form

$$
\frac{d}{dt}(\pi_t, \chi_t) = H_t(\pi_t, \chi_t) + (\alpha_t, 0)
$$

where $H_t : \mathcal{M}_{B'} \times \mathbb{R} \to \mathcal{M}_{B'} \times \mathbb{R}$ is linear, $H_t(\pi, \chi) \geq 0$ whenever $(\pi, \chi) \geq 0$, where $\alpha_t \in \mathcal{M}_{B'}$ with $\alpha_t \geq 0$, and where we have estimates, for $t \leq 1$,

$$
\|H_t(\pi, \chi)\| \leq C\|(\pi, \chi)\| \qquad \text{and} \qquad \|\alpha_t\| \leq C
$$

for some constant $C < \infty$ depending only on φ and B'. Therefore we can apply the same sort of argument that we used for non-negativity to see that $\pi_t \geq 0$ and $\chi_t \geq 0$ for all $t \leq 1$, and then for all $t < \infty$, as required. Explicitly, H_t is given by

$$
\begin{aligned}
\langle (f, a), H_t(\pi, \chi)\rangle &= \frac{1}{2}\int_{E\times E} f\theta_t(x+y)1_{x+y\in B}K(x,y) \\
&\qquad\qquad \times (\theta_t(x)^{-1}\pi(dx)\mu_t^{B'}(dy) + \theta_t(y)^{-1}\mu_t^B(dx)\pi(dy)) \\
&\quad + \int_{E\times E} f\theta_t(x)(\varphi(x)\varphi(y) - K(x,y))\mu_t^{B'}(dx)\theta_t(y)^{-1}\pi(dy) \\
&\quad + \chi \int_E f\theta_t(x)\varphi(x)\mu_t^{B'}(dx) \\
&\quad + \frac{1}{2}a\int_{E\times E}\{\varphi(x)+\varphi(y)-\varphi(x+y)\}K(x,y) \\
&\qquad\qquad (\theta_t(x)^{-1}\pi(dx)\mu_t^{B'}(dy) + \theta_t(y)^{-1}\mu_t^B(dx)\pi(dy))
\end{aligned}
$$

and α_t is given by

$$
\langle f, \alpha_t\rangle = \frac{1}{2}\int_{E\times E} f\theta_t(x+y)1_{x+y\in B'\setminus B}K(x,y)\mu_t^{B'}(dx)\mu_t^{B'}(dy).
$$

□

Proposition 2.5. *Suppose that $(\nu_t)_{t<T}$ is a local solution of the coagulation equation (2.1), starting from μ_0. Then, for all compact sets $B \subseteq E$ and all $t < T$,*

$$\mu_t^B \leq \nu_t, \quad \langle \varphi, \mu_t^B \rangle + \lambda_t^B \geq \langle \varphi, \nu_t \rangle.$$

Proof. Set

$$\begin{aligned}
\theta_t(x) &= \exp \int_0^t \left(\int_E K(x,y)\mu_s^B(dy) + \lambda_s^B \varphi(x) \right) ds, \\
\nu_t^B &= 1_B \nu_t, \\
\pi_t &= \theta_t(\nu_t^B - \mu_t^B), \\
\chi_t &= \langle \varphi, \mu_t^B \rangle + \lambda_t^B - \langle \varphi, \nu_t \rangle.
\end{aligned}$$

We have to show that $\pi_t \geq 0$ and $\chi_t \geq 0$. By an obvious modification of Proposition 2.3 we have, for all bounded measurable functions f,

$$\frac{d}{dt}\langle f, \pi_t \rangle = \langle f\, \partial\theta/\partial t, \nu_t^B - \mu_t^B \rangle + \langle f\theta_t 1_B, L(\nu_t) \rangle - \langle (f\theta_t, 0), L^B(\mu_t^B, \lambda_t^B) \rangle.$$

By (2.3), we have

$$\langle \varphi, \nu_t \rangle \leq \langle \varphi, \nu_0 \rangle + \frac{1}{2}\int_0^t \int_{E\times E} \{\varphi(x+y) - \varphi(x) - \varphi(y)\} K(x,y)\nu_s(dx)\nu_s(dy)\, ds.$$

On the other hand

$$\frac{d}{dt}(\langle \varphi, \mu_t^B \rangle + \lambda_t^B) = \frac{1}{2}\int_{E\times E} \{\varphi(x+y) - \varphi(x) - \varphi(y)\} K(x,y)\mu_t^B(dx)\mu_t^B(dy).$$

So $\chi_t \geq \rho_t$ where

$$\rho_t = \frac{1}{2}\int_0^t \int_{E\times E} \{\varphi(x) + \varphi(y) - \varphi(x+y)\} K(x,y)(\nu_s(dx)\nu_s(dy) - \mu_s^B(dx)\mu_s^B(dy))\, ds.$$

Now $(\pi_t, \rho_t) \in \mathcal{M}_B \times \mathbb{R}$ obeys a differential equation of the form

$$\frac{d}{dt}(\pi_t, \rho_t) = H_t(\pi_t, \rho_t) + (\alpha_t, \beta_t)$$

where $H_t : \mathcal{M}_B \times \mathbb{R} \to \mathcal{M}_B \times \mathbb{R}$ is linear, $H_t(\pi, \rho) \geq 0$ whenever $(\pi, \rho) \geq 0$, where $\alpha_t \geq 0$, $\beta_t \geq 0$ and we have estimates of the form

$$\|H_t(\pi, \rho)\| \leq C\|(\pi, \rho)\|, \|\alpha_t\| \leq C, |\beta_t| \leq C.$$

It follows that $\pi_t \geq 0$ and $\rho_t \geq 0$, so also $\chi_t \geq 0$ as required. Explicitly, H_t is given by

$$
\begin{aligned}
\langle (f,a), H_t(\pi,\rho)\rangle = \frac{1}{2}\int_{E\times E} & f\theta_t(x+y)1_{x+y\in B}K(x,y) \\
& \times (\theta_t(x)^{-1}\pi(dx)\nu_t^B(dy) + \theta_t(y)^{-1}\mu_t^B(dx)\pi(dy)) \\
+ \int_{E\times E} & f\theta_t(x)(\varphi(x)\varphi(y) - K(x,y))\theta_t(y)^{-1}\nu_t^B(dx)\pi(dy) \\
+ \rho\int_E & f\theta_t(x)\varphi(x)\nu_t^B(dx) \\
+ \frac{1}{2}a\int_{E\times E} & \{\varphi(x)+\varphi(y)-\varphi(x+y)\}K(x,y) \\
& \times (\theta_t(x)^{-1}\pi(dx)\nu_t^B(dy) + \theta_t(y)^{-1}\mu_t^B(dx)\pi(dy))
\end{aligned}
$$

and α_t, β_t are given by

$$
\begin{aligned}
\langle f, \alpha_t\rangle &= \int_{E\times E} f\theta_t(x)(\varphi(x)\varphi(y) - K(x,y))1_{y\notin B}\nu_t^B(dx)\nu_t(dy) \\
&\quad + (\chi_t - \rho_t)\int_E f\theta_t(x)\varphi(x)\nu_t^B(dx) \\
&\quad + \frac{1}{2}\int_{E\times E} f\theta_t(x+y)1_{x+y\in B}K(x,y)1_{(x,y)\notin B\times B}\nu_t(dx)\nu_t(dy), \\
\beta_t &= \frac{1}{2}\int_{E\times E}\{\varphi(x)+\varphi(y)-\varphi(x+y)\}K(x,y)1_{(x,y)\notin B\times B}\nu_t(dx)\nu_t(dy). \qquad \square
\end{aligned}
$$

This concludes the proof of Theorem 2.1.

3. An example of non-uniqueness

We construct in this section an example of Smoluchowski's coagulation equation having at least two solutions, both of which are moreover conservative.

Consider the system of differential equations

$$
\frac{d}{dt}m_n(t) = -\lambda_n m_n(t)m_{n+1}(t), \quad n = 1,2,\ldots \tag{3.1}
$$

For any solution we have

$$
m_n(t) = m_n(0)\exp\{-\lambda_n\int_0^t m_{n+1}(s)\,ds\}.
$$

Assume that $m_n(0) \geq 0$ for all n, then $m_n(t) \geq 0$ for all n and t. Note that, if m_N is given, and we consider the system restricted to $n \leq N-1$, then m_n is decreasing in m_N when $N-n$ is odd, and increasing in m_N when $N-n$ is even.

Fix N and consider the case

$$m_n(0) = \begin{cases} 2^{-n}, & n = 1, \dots, 2N \\ 0, & n = 2N+1, \dots \end{cases}$$

and

$$\lambda_n = 8^n \quad \text{for all } n.$$

Proposition 3.1. *We have for all n and t*

$$m_{2n}(t) \geq \frac{1}{2} m_{2n}(0),$$
$$m_{2n+1}(t) \leq m_{2n+1}(0) \exp\{-4^{2n} t\}.$$

Proof. Certainly $m_{2N}(t) = m_{2N}(0) \geq \frac{1}{2} m_{2N}(0)$. Suppose that $n < N$ and

$$m_{2n+2}(t) \geq \frac{1}{2} m_{2n+2}(0) \quad \text{for all } t.$$

Then

$$\begin{aligned} m_{2n+1}(t) &\leq m_{2n+1}(0) \exp\{-\lambda_{2n+1} m_{2n+2}(0) t/2\} \\ &= m_{2n+1}(0) \exp\{-4^{2n} t\} \end{aligned}$$

so

$$\int_0^\infty m_{2n+1}(t)\, dt \leq m_{2n+1}(0) 4^{-2n} = \frac{1}{2}\, 8^{-2n}$$

so

$$m_{2n}(t) \geq m_{2n}(0) \exp\{-8^{-2n} \lambda_{2n}/2\} \geq \frac{1}{2} m_{2n}(0).$$

Hence the proposition follows by induction. □

We denote the solution just considered by m^{2N}. The same arguments establish corresponding inequalities for m^{2N+1}, where the roles of even and odd are swapped. Now we let $N \to \infty$. For $N \geq n$, $m^{2N}_{2n}(t)$ is decreasing in N and $m^{2N}_{2n+1}(t)$ is increasing in N for all n. We set

$$m^+_n(t) = \lim_{N \to \infty} m^{2N}_n(t).$$

The integral equation

$$m^{2N}_n(t) = m^{2N}_n(0) - \lambda_n \int_0^t m^{2N}_n(s) m^{2N}_{n+1}(s)\, ds$$

passes to the limit to give

$$m^+_n(t) = m^+_n(0) - \lambda_n \int_0^t m^+_n(s) m^+_{n+1}(s)\, ds$$

so m^+ is differentiable with

$$\frac{d}{dt} m^+_n(t) = -\lambda_n m^+_n(t) m^+_{n+1}(t).$$

So $(m_n^+ : n \geq 1)$ solves the original system of equations. The same argument produces another solution $(m_n^- : n \geq 1)$ given by

$$m_n^-(t) = \lim_{N\to\infty} m_n^{2N+1}(t).$$

We have $m_n^+(0) = m_n^-(0) = 2^{-n}$ for all n. But

$$m_{2n}^+(t) \geq \frac{1}{2} m_{2n}^+(0),$$
$$m_{2n}^-(t) \leq \exp\{-4^{2n-1}t\} m_{2n}^-(0)$$

for all n and t, so $m^+ \neq m^-$.

We now use the solutions m^+ and m^- to construct an example of Smoluchowski's coagulation equation having at least two conservative solutions. Let $x_1, x_2, \dots$ be an increasing sequence in $(0,\infty)$ which is linearly independent over $\mathbb{Z}$. For

$$x = x_n + (k_1 x_1 + \ldots + k_{n-1}x_{n-1}), \quad k_1, \dots, k_{n-1} \in \mathbb{Z}$$

we write $n(x) = n$. Denote by I the set of all such x and define $n(x) = 0$ if $x \notin I$. Define $K : E \times E \to [0,\infty)$ by

$$K(x,y) = \begin{cases} \lambda_n & \text{if } \{n(x), n(y)\} = \{n, n+1\} \text{ and } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mu_0 = \sum_{n=1}^{\infty} \varepsilon_{x_n} 2^{-n}$$

and consider Smoluchowski's coagulation equation

$$\frac{d}{dt}\langle f, \mu_t\rangle = \frac{1}{2}\int_{E\times E} \{f(x+y) - f(x) - f(y)\} K(x,y)\mu_t(dy)\mu_t(dx)$$

starting from μ_0.

According to the definition made in §2, for a solution, we require

$$\int_0^t \int_{B\times E} K(x,y)\mu_s(dx)\mu_s(dy)\, ds < \infty$$

for all t and all compact sets $B \subseteq E$, and, for all bounded measurable functions f of compact support

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \frac{1}{2}\int_0^t \int_{E\times E} \{f(x+y) - f(x) - f(y)\} K(x,y)\mu_s(dx)\mu_s(dy)\, ds.$$

Consider for $n = 1, 2, \dots,$

$$m_n(t) = \mu_t(\{x : n(x) = n\}).$$

Take $f(x) = 1_{n(x)=n, x\leq k}$ and let $k \to \infty$ to obtain

$$\frac{d}{dt} m_n(t) = -\lambda_n m_n(t) m_{n+1}(t).$$

We deduce that any solution $(\mu_t)_{t\geq 0}$ of the coagulation equation gives rise to a solution $(m_n(t) : n \geq 1)_{t\geq 0}$ of the system (3.1). On the other hand, for any solution $(m_n(t) : n \geq 1)_{t\geq 0}$ of this system, we obtain a solution $(\mu_t)_{t\geq 0}$ of the coagulation equation by

$$\frac{d}{dt}\mu_t(\{x\}) = -(\lambda_{n-1}m_{n-1}(t) + \lambda_n m_{n+1}(t))\mu_t(\{x\}) + \frac{1}{2}\lambda_{n-1}\sum_{\substack{y,z\in I\\ y+z=x}}\mu_t(\{y\})\mu_t(\{z\})$$

whenever $n(x) = n$. Hence the coagulation equation has two distinct solutions $(\mu_t^+)_{t\geq 0}$ and $(\mu_t^-)_{t\geq 0}$ corresponding to $(m_t^+)_{t\geq 0}$ and $(m_t^-)_{t\geq 0}$. We now show these solutions are conservative. The idea of the proof is to show that the proportion of original particles making at least $2n$ jumps falls off geometrically in n.

Proposition 3.2. *Suppose that μ_0 has finite total mass density. Then the solution $(\mu_t^+)_{t\geq 0}$ is conservative.*

Proof. For $x = k_1x_1 + \ldots + k_{n-1}x_{n-1} + x_n \in I$, define

$$k_m(x) = \begin{cases} k_m & \text{if } m < n \\ 1 & \text{if } m = n \\ 0 & \text{if } m > n. \end{cases}$$

Note that $k_m : I \to \mathbb{Z}^+$ is additive. For $m \leq n$ consider

$$v_{m,n}(t) = \int_I k_m(x)1_{n(x)\leq n}\mu_t(dx).$$

Then $v_{m,n}(t)$ is non-increasing in t. Note that

$$\int_{I\times I} k_m(x)1_{n(x)\leq n}K(x,y)\mu_t(dx)\mu_t(dy) \leq 8^n v_{m,n}(t) < \infty$$

so, by dominated convergence, since $(\mu_t^+)_{t\geq 0}$ is a solution,

$$\frac{d}{dt}v_{m,n}(t) = -\lambda_n m_{n+1}(t)\int_I k_m(x)1_{n(x)=n}\mu_t(dx).$$

Note also

$$\frac{d}{dt}\int_I k_m(x)1_{n(x)=n}\mu_t(dx) \leq -\frac{d}{dt}v_{m,n-1}(t)$$

so

$$\int_I k_m(x)1_{n(x)=n}\mu_t(dx) \leq r_{m,n-1}(t)$$

where $r_{m,n}(t) = 2^{-m} - v_{m,n}(t)$. Now for n even

$$r_{m,n}(t) \leq r_{m,n-1}(t)\int_0^t \lambda_n m_{n+1}(s)\,ds \leq \frac{1}{2}r_{m,n-1}(t)$$

so $r_{m,n}(t) \to 0$ as $n \to \infty$. Hence $v_{m,n}(t) \to 2^{-m}$ as $n \to \infty$ and hence

$$\int_I x\mu_t(dx) = \sum_m x_m \int_I k_m(x)\mu_t(dx) = \sum_m x_m 2^{-m} = \int_I x\mu_0(dx).$$

□

We make some remarks on the relation between this example and the results of §2. The construction of the example makes it insensitive to the additive structure of E. We require very little of the sequence $(x_n)_{n\geq 1}$. By taking $x_n \approx 8^n$ we can satisfy the condition

$$K(x,y) \leq x$$

but we get

$$\int_E x\mu_0(dx) = \infty.$$

On the other hand, by taking $x_n \approx \alpha^n$, for some $\alpha < 2$, we get

$$\int_E x\mu_0(dx) < \infty$$

but the relation $K(x,y) \leq Cxy$ for all $x, y \in E$ does not hold for any $C < \infty$. Thus, however we choose $(x_n)_{n\geq 1}$, we cannot regard $(\mu_t^{\pm})_{t\geq 0}$ as a strong solution, even in small time. This is of course implied also by the uniqueness of strong solutions established in §2.

It would be nice to find an example of this type where the initial mass distibution is supported on $\mathbb{N}$. It may be that for integers $x_n \to \infty$ sufficiently fast, the analogous equation exhibits the same sort of behaviour. However we have not established whether this is true.

4. Hydrodynamic limit for the stochastic coalescent

In this section we shall prove some limit theorems for the stochastic coalescent. There are two main results. In Theorem 4.1, generalizing a result of Jeon [Jeo], we prove a tightness result for the stochastic coalescent, which implies a general existence theorem for solutions of Smoluchowski's equation. Then, in Theorem 4.4, we prove weak convergence of the stochastic coalescent to any strong solution of Smoluchowski's equation. The methods used are mostly standard tools from the theory of weak convergence on Skorohod spaces. The problem-specific idea which leads to Theorem 4.4 is the construction of a coupled family of particle systems, converging to the stochastic coalescent, in direct analogy with the method of §2. A version of this idea was also discovered independently by Kurtz [Kur]. The case of a discrete mass distribution may also be treated using a differential equation approach instead of weak convergence: this is simpler and more effective, establishing convergence at an exponential rate in the number of particles. The particle system we consider has been considered, in various special cases, by many others. In particular, it was considered in full generality by Marcus [Mar68] and Lushnikov [Lus78]. Recall that $E = (0,\infty)$ and that the coagulation kernel K is a symmetric measurable function $K : E \times E \to [0,\infty)$.

Let X_0 be a finite, integer-valued measure on E. We can write X_0 as a sum of unit masses

$$X_0 = \sum_{i=1}^{m} \varepsilon_{x_i}$$

for some $x_1, \dots, x_m \in E$. We think of X_0 as representing a system of m particles, labelled by their masses $x_1, \dots, x_m$. A Markov process $(X_t)_{t\geq 0}$ of finite, integer-valued measures on E can be constructed as follows: for each pair $i<j$, take an independent exponential random time T_{ij} of parameter $K(x_i, x_j)$ and set $T = \min_{i<j} T_{ij}$; set $X_t = X_0$ for $t < T$ and set

$$X_T = X_0 - \varepsilon_{x_i} - \varepsilon_{x_j} + \varepsilon_{x_i+x_j} \quad \text{ if } T = T_{ij};$$

then begin the construction afresh from X_T. In this process, each pair of particles $\{x_i, x_j\}$ coalesces at rate $K(x_i, x_j)$ to form a new particle $x_i + x_j$. We call $(X_t)_{t\geq 0}$ a *stochastic coalescent with coagulation kernel* K.

Denote by d some metric on $\mathcal{M}$ which is compatible with the topology of weak convergence, that is to say $d(\mu_n, \mu) \to 0$ if and only if $\langle f, \mu_n\rangle \to \langle f, \mu\rangle$ for all bounded continuous functions $f : E \to \mathbb{R}$. We choose d so that $d(\mu, \mu') \leq \|\mu - \mu'\|$ for all $\mu, \mu' \in \mathcal{M}$. When the class of functions f is restricted to those of bounded support we get a weaker topology, also metrizable, and we denote by d_0 some compatible metric, with $d_0 \leq d$.

The following result is a first attempt at proving weak convergence for the stochastic coalescent. It is less than satisfactory because it does not enable us to show uniqueness of limits. We include it here, partly as a warm-up for the more intricate arguments used later, and partly because it provides the best result on global existence of solutions to Smoluchowski's equation that we know. A version of the result where μ_0 is supported on $\mathbb{N}$ and where $\varphi(x) = x$ has been proved already by Jeon [Jeo].

Theorem 4.1. *Let $K : E \times E \to [0,\infty)$ be a symmetric continuous function and let μ_0 be a measure on E. Assume that, for some continuous sublinear function $\varphi : E \to (0,\infty)$,*

$$K(x,y) \leq \varphi(x)\varphi(y), \quad \text{ for all } x, y \in E,$$

$$\varphi(x)^{-1}\varphi(y)^{-1}K(x,y) \to 0, \quad \text{ as } (x,y) \to \infty.$$

Assume also that $\langle \varphi, \mu_0\rangle < \infty$. Let $(X_t^n)_{t\geq 0}$ be a sequence of stochastic coalescents, with coagulation kernel K. Set $\tilde{X}_t^n = n^{-1}X^n_{n^{-1}t}$ and suppose that

$$d_0(\varphi\tilde{X}_0^n, \varphi\mu_0) \to 0$$

as $n \to \infty$ and that, for some constant $\Lambda < \infty$, for all n

$$\langle \varphi, \tilde{X}_0^n\rangle \leq \Lambda.$$

Then the sequence of laws of $\tilde{X}^n$ on $D([0,\infty),(\mathcal{M},d_0))$ is tight. Moreover, for any weak limit point X, almost surely, $(X_t)_{t\geq 0}$ is a solution of Smoluchowski's coagulation equation (2.1)*. In particular, this equation has at least one solution.*

Proof. For an integer-valued measure μ on E, denote by $\mu^{(1)}$ the integer-valued measure on $E \times E$ given by

$$\mu^{(1)}(A \times A') = \mu(A)\mu(A') - \mu(A \cap A').$$

(This is simply the counting measure for ordered pairs of masses of distinct particles.) Similarly, when $n\mu$ is an integer-valued measure, set

$$\mu^{(n)}(A \times A') = \mu(A)\mu(A') - n^{-1}\mu(A \cap A')$$

and note that $n^2\mu^{(n)} = (n\mu)^{(1)}$. For a bounded measurable function f on E, set

$$\begin{aligned} L^{(n)}(\mu)(f) &= \langle f, L^{(n)}(\mu)\rangle \\ &= \frac{1}{2}\int_{E\times E}\{f(x+y)-f(x)-f(y)\}K(x,y)\mu^{(n)}(dx,dy), \\ Q^{(n)}(\mu)(f) &= \frac{1}{2}\int_{E\times E}\{f(x+y)-f(x)-f(y)\}^2K(x,y)\mu^{(n)}(dx,dy). \end{aligned}$$

Then

$$M_t^{f,n} = \langle f, X_t^n\rangle - \langle f, X_0^n\rangle - \int_0^t L^{(1)}(X_s^n)(f)\,ds$$

is a martingale, having previsible increasing process

$$\langle M^{f,n}\rangle_t = \int_0^t Q^{(1)}(X_s^n)(f)\,ds.$$

Set $\tilde{M}_t^{f,n} = n^{-\frac{1}{2}}M_{n^{-1}t}^{f,n}$, then we have

$$\langle f, \tilde{X}_t^n\rangle = \langle f, \tilde{X}_0^n\rangle + n^{-\frac{1}{2}}\tilde{M}_t^{f,n} + \int_0^t L^{(n)}(\tilde{X}_s^n)(f)\,ds, \tag{4.1}$$

$$\langle \tilde{M}^{f,n}\rangle_t = \int_0^t Q^{(n)}(\tilde{X}_s^n)(f)\,ds.$$

Since φ is subadditive, we have $\langle \varphi, \tilde{X}_t^n\rangle \leq \Lambda$ for all n and t. Hence by (2.4)

$$\begin{aligned} |L^{(n)}(\tilde{X}_t^n)(f)| &\leq 2\|f\|\Lambda^2, \\ Q^{(n)}(\tilde{X}_t^n)(f) &\leq 4\|f\|^2\Lambda^2. \end{aligned}$$

Assume that $|f| \leq \varphi \wedge 1$. Then

$$|\langle f, \tilde{X}_t^n\rangle| \leq \Lambda$$

for all t, so we have compact containment. Moreover, by Doob's L^2-inequality, for all $s < t$,

$$\mathbb{E} \sup_{s \le r \le t} |\tilde{M}_r^{f,n} - \tilde{M}_s^{f,n}|^2 \le 4\mathbb{E} \int_s^t Q^{(n)}(\tilde{X}_r^n)(f) dr \le 16\Lambda^2(t-s)$$

so

$$\mathbb{E} \sup_{s \le r \le t} |\langle f, \tilde{X}_r^n - \tilde{X}_s^n \rangle|^2 \le C\{(t-s)^2 + n^{-1}(t-s)\}$$

where $C < \infty$ depends only on Λ. Hence, by a standard tightness criterion, the laws of the sequence $\langle f, \tilde{X}^n \rangle$ on $D([0,\infty), \mathbb{R})$ are tight. See for example [EK86], Corollary 7.4. We note the bound

$$\|(\varphi \wedge 1)\tilde{X}_t^n\| \le \langle \varphi, \tilde{X}_t^n \rangle \le \Lambda$$

for all t. Hence we can apply Jakubowski's criterion [Jak86] to see that the laws of the sequence $(\varphi \wedge 1)\tilde{X}^n$ on $D([0,\infty), \mathcal{M}_{[0,\infty]})$ are tight. By consideration of subsequences and a theorem of Skorohod, see, for example, Pollard [Pol84], Chapter IV, it suffices from this point on to consider the case where $(\varphi \wedge 1)\tilde{X}^n$ converges almost surely in $D([0,\infty), \mathcal{M}_{[0,\infty]})$, with limit $(\varphi \wedge 1)X$ say. We denote also by X the process in $\mathcal{M}_E$ obtained by restriction of measures. Note that

$$\|\tilde{X}_t^n - \tilde{X}_{t-}^n\| \le 3/n$$

so $X \in C([0,\infty), \mathcal{M}_E)$. Moreover $\varphi^\delta = \varphi 1_{(0,\delta]}$ is subadditive, so

$$\langle \varphi^\delta, \tilde{X}_t^n \rangle \le \langle \varphi^\delta, \tilde{X}_0^n \rangle \le \langle \varphi^\delta, \mu_0 \rangle + |\langle \varphi^\delta, \tilde{X}_0^n - \mu_0 \rangle|$$

and so, given $\varepsilon > 0$, we can find $\delta > 0$ so that

$$\sup_n \sup_t \langle \varphi^\delta, \tilde{X}_t^n \rangle < \varepsilon.$$

Given a continuous bounded function $f : E \to \mathbb{R}$ of bounded support, we can write $f = f_1 + f_2$ where f_1 is continuous of compact support and f_2 is supported in $(0,\delta)$ with $\|f_2\| \le \|f\|$. Then

$$\limsup_{n\to\infty} \sup_{s\le t} \langle \varphi f, \tilde{X}_s^n - X_s \rangle \le \lim_{n\to\infty} \sup_{s\le t} \langle \varphi f_1, \tilde{X}_s^n - X_s \rangle + \|f\| \sup_n \sup_s \langle \varphi^\delta, \tilde{X}_s^n \rangle \le \varepsilon \|f\|.$$

Since f and ε were arbitrary, this shows that

$$\sup_{s\le t} d_0(\varphi \tilde{X}_s^n, \varphi X_s) \to 0 \quad \text{a.s.} \tag{4.2}$$

We now wish to pass to the limit in (4.1). Let us suppose for now that $f : E \to \mathbb{R}$ is continuous and of compact support B. Then, as $n \to \infty$

$$\begin{aligned}
\mathbb{E}(\sup_{s\le t} |n^{-\frac{1}{2}}\tilde{M}^{f,n}_s|^2) &\le \frac{4}{n}\mathbb{E}\langle \tilde{M}^{f,n}\rangle_t \le \frac{16\Lambda^2\|f\|^2}{n} \to 0,\\
|(L - L^{(n)})(\tilde{X}^n_s)(f)| &= \frac{1}{n}\left|\int_E \{f(2x) - 2f(x)\}K(x,x)\tilde{X}^n_s(dx)\right|\\
&\le \frac{3\|f\|}{n}\int_{B\cup 2B} \varphi(x)^2\tilde{X}^n_s(dx)\\
&\le \frac{3\|f\|}{n}\|\varphi 1_{B\cup 2B}\|\langle\varphi, \tilde{X}^n_0\rangle \to 0.
\end{aligned}$$

Hence it will suffice to show that as $n \to \infty$

$$\sup_{s\le t} |\langle f, L(\tilde{X}^n_s) - L(X_s)\rangle| \to 0 \quad \text{a.s.} \tag{4.3}$$

where we recall that

$$\langle f, L(\mu)\rangle = \frac{1}{2}\int \{f(x+y) - f(x) - f(y)\}K(x,y)\mu(dx)\mu(dy).$$

Given $\delta > 0$ and $N < \infty$, we can write $K = K_1 + K_2$, where K_1 is continuous of compact support and where $0 \le K_2 \le K$ and K_2 is supported on

$$F_1 \cup F_2 \cup F_2 = \{(x,y) : x \le \delta\} \cup \{(x,y) : y \le \delta\} \cup \{(x,y) : |(x,y)| \ge N\}.$$

Then, with an obvious notation, by (4.2)

$$\sup_{s\le t} |\langle f, L_1(\tilde{X}^n_s) - L_1(X_s)\rangle| \to 0 \quad \text{a.s.}$$

whereas, for K_2, we use the estimates

$$\begin{aligned}
\|K1_{F_1}\mu\otimes\mu\| = \|K1_{F_2}\mu\otimes\mu\| &\le \langle\varphi,\mu\rangle\langle\varphi^\delta,\mu\rangle,\\
\|K1_{F_3}\mu\otimes\mu\| &\le \beta_N\langle\varphi,\mu\rangle^2,
\end{aligned}$$

where $\beta_N = \sup_{|(x,y)|\ge N}\varphi(x)^{-1}\varphi(y)^{-1}K(x,y)$. Now,

$$\langle\varphi, \tilde{X}^n_t\rangle \le \langle\varphi, \tilde{X}^n_0\rangle, \quad \langle\varphi^\delta, \tilde{X}^n_t\rangle \le \langle\varphi^\delta, \tilde{X}^n_0\rangle$$

and, given $\varepsilon > 0$, we can find δ and N so that

$$\begin{aligned}
\langle\varphi^\delta, \tilde{X}^n_0\rangle &\le \tfrac{1}{3}\varepsilon\Lambda^{-1}, \quad \text{for all } n,\\
\langle\varphi^\delta, \mu_0\rangle &\le \tfrac{1}{3}\varepsilon\Lambda^{-1},\\
\beta_N &\le \tfrac{1}{3}\varepsilon\Lambda^{-2}.
\end{aligned}$$

Then

$$|\langle f, L_2(\tilde{X}^n_t)\rangle| \le \varepsilon, \quad |\langle f, L_2(X_t)\rangle| \le \varepsilon$$

for all n and t. Hence

$$\limsup_{n\to\infty} \sup_{s\le t} |\langle f, L(\tilde{X}^n_s) - L(X_s)\rangle| \le 2\varepsilon.$$

But ε was arbitrary, so (4.3) is proved. Hence we can let $n \to \infty$ in (3.1) to obtain

$$\langle f, X_t\rangle = \langle f, X_0\rangle + \int_0^t \langle f, L(X_s)\rangle\, ds$$

for all continuous functions $f : E \to \mathbb{R}$ of compact support. By using the bounds (2.4) and $\langle \varphi, X_t\rangle \le \Lambda$, and a straightforward limit argument, we can extend this equation to all bounded measurable functions f. In particular, almost surely, X is a solution of Smoluchowski's equation, in the sense of §2. $\square$

A corollary of Theorem of 4.1 is that, whenever we know Smoluchowski's equation has at most one solution, then, under the hypotheses of Theorem 4.1, we can deduce, for all t

$$\sup_{s\le t} d_0(\tilde{X}^n_s, \mu_s) \to 0$$

in probability as $n \to \infty$, for the solution $(\mu_t)_{t\ge 0}$ provided by Theorem 4.1. However, we can only prove uniqueness of solutions in the presence of a strong solution. So we prefer to formulate our main limit result, Theorem 4.4, in that context, when a new approach allows certain other hypotheses to be relaxed.

For the remainder of this section we will assume that we have chosen a continuous sublinear function $\varphi : E \to (0, \infty)$ and that K satisfies

$$K(x, y) \le \varphi(x)\varphi(y) \quad \text{for all } x, y \in E. \tag{4.4}$$

Our further analysis of the stochastic coalescent will rest on an approximation by a coupled family of Markov processes $(X^B_t, \Lambda^B_t)_{t\ge 0}$, indexed by sets $B \subseteq E$ which we now describe. Each process $(X^B_t)_{t\ge 0}$ will take values in the finite integer-valued measures on E, whereas $(\Lambda^B_t)_{t\ge 0}$ will be a non-decreasing process in $[0, \infty)$. Let us suppose given initial values (X^B_0, Λ^B_0), for all B, such that X^B_0 is supported in B and such that, whenever $B \subseteq B'$

$$X^B_0 \le X^{B'}_0, \quad \langle \varphi, X^B_0\rangle + \Lambda^B_0 \ge \langle \varphi, X^{B'}_0\rangle + \Lambda^{B'}_0.$$

Write $X_0 = X^E_0$ as a sum of unit masses

$$X_0 = \sum_{i=1}^m \varepsilon_{x_i}$$

where $x_1, \dots, x_m \in E$. There is a unique increasing map

$$B \mapsto I(B) \subseteq \{1, \dots, m\}$$

such that

$$X^B_0 = \sum_{i\in I(B)} \varepsilon_{x_i}.$$

Set

$$\nu^B = \Lambda_0^B - \sum_{j \notin I(B)} \varphi(x_j).$$

Note that ν^B decreases as B increases and that $\nu^E = \Lambda_0^E \geq 0$. For $i < j$, take independent exponential random variables T_{ij} of parameter $K(x_i, x_j)$. Set $T_{ji} = T_{ij}$. Also, for $i \neq j$, take independent exponential random variables S_{ij} of parameter $\varphi(x_i)\varphi(x_j) - K(x_i, x_j)$. We can construct, independently for each i, a family of independent exponential random variables S_i^B, increasing in B, with S_i^B having parameter $\varphi(x_i)\nu^B$. Set

$$T_i^B = \min_{j \notin I(B)} (T_{ij} \wedge S_{ij}) \wedge S_i^B$$

and note that T_i^B is an exponential random variable of parameter

$$\sum_{j \notin I(B)} \varphi(x_i)\varphi(x_j) + \varphi(x_i)\nu^B = \varphi(x_i)\Lambda_0^B.$$

For each B, the random variables

$$(T_{ij}, T_i^B : i, j \in I(B), i < j)$$

form an independent family, whereas, for $i \in I(B)$ and $j \notin I(B)$, we have

$$T_i^B \leq T_{ij}$$

and for $B \subseteq B'$ and all i we have

$$T_i^B \leq T_i^{B'}.$$

Now set

$$T = (\min_{i<j} T_{ij}) \wedge (\min_i T_i^{\emptyset}).$$

We set $(X_t^B, \Lambda_t^B) = (X_0^B, \Lambda_0^B)$ for $t < T$ and set

$$(X_t^B, \Lambda_T^B) = \begin{cases} (X_0^B - \varepsilon_{x_i} - \varepsilon_{x_j} + \varepsilon_{x_i+x_j}, \Lambda_0^B) & \text{if } T = T_{ij}, i, j \in I(B), x_i + x_j \in B \\ (X_0^B - \varepsilon_{x_i} - \varepsilon_{x_j}, \Lambda_0^B + \varphi(x_i + x_j)) & \text{if } T = T_{ij}, i, j \in I(B), x_i + x_j \notin B \\ (X_0^B - \varepsilon_{x_i}, \Lambda_0^B + \varphi(x_i)) & \text{if } T = T_i^B, i \in I(B) \\ (X_0^B, \Lambda_0^B) & \text{otherwise.} \end{cases}$$

It is straightforward to check that X_T^B is supported on B and, for $B \subseteq B'$

$$X_T^B \leq X_T^{B'}, \quad \langle \varphi, X_T^B \rangle + \Lambda_T^B \geq \langle \varphi, X_T^{B'} \rangle + \Lambda_T^{B'}.$$

Hence we can repeat the above construction independently from time T, to obtain a family of Markov processes $(X_t^B, \Lambda_t^B)_{t \geq 0}$ such that X_t^B is supported on B and for $B \subseteq B'$ and all t

$$X_t^B \leq X_t^{B'}, \quad \langle \varphi, X_t^B \rangle + \Lambda_t^B \geq \langle \varphi, X_t^{B'} \rangle + \Lambda_t^{B'}. \tag{4.5}$$

At the outset, we assumed that both X_0^B and Λ_0^B were given, for all B. From now on we shall suppose simply that $X_0 = X_0^E$ is given and take

$$X_0^B = 1_B X_0, \quad \Lambda_0^B = \langle \varphi 1_{B^c}, X_0 \rangle.$$

Of course these relations do not remain valid as time evolves.

For each fixed B, the process $(X_t^B, \Lambda_t^B)_{t \geq 0}$ may be regarded as a finite state-space Markov chain having three sorts of transition. Each pair of particles x_i, x_j in X_t^B is, at rate $K(x_i, x_j)$, removed; if $x_i + x_j \in B$, the merged particle is added to X_t^B, if not, $\varphi(x_i + x_j)$ is added to Λ_t^B. Also, each particle x_i in X_t^B is, at rate $\varphi(x_i)\Lambda_t^B$, removed and $\varphi(x_i)$ added to Λ_t^B. In particular, for the choice of initial values made above, $\Lambda_t^E = 0$ for all t and $X_t = X_t^E$ is simply the stochastic coalescent with coagulation kernel K with which we began.

We now proceed to identify some martingales associated with $(X_t^B, \Lambda_t^B)_{t \geq 0}$. Recall that, when $n\mu$ is an integer-valued measure on E, we denote by $\mu^{(n)}$ the measure on $E \times E$ characterized by

$$\mu^{(n)}(A \times A') = \mu(A)\mu(A') - n^{-1}\mu(A \cap A').$$

Given an integer-valued measure μ on E and given $\lambda \geq 0$, define for any bounded measurable function f on E and for $a \in \mathbb{R}$

$$\begin{aligned} &L^{B,(1)}(\mu, \lambda)(f, a) = \langle (f, a), L^{B,(1)}(\mu, \lambda) \rangle = \\ &\frac{1}{2} \int_{E \times E} \{ f(x+y) 1_{x+y \in B} + a\varphi(x+y) 1_{x+y \notin B} - f(x) - f(y) \} K(x,y) \mu^{(1)}(dx, dy) \\ &\qquad\qquad\qquad\qquad + \lambda \int_E \{ a\varphi(x) - f(x) \} \varphi(x) \mu(dx) \end{aligned}$$

and

$$\begin{aligned} &Q^{B,(1)}(\mu, \lambda)(f, a) = \\ &\frac{1}{2} \int_{E \times E} \{ f(x+y) 1_{x+y \in B} + a\varphi(x+y) 1_{x+y \notin B} - f(x) - f(y) \}^2 K(x,y) \mu^{(1)}(dx, dy) \\ &\qquad\qquad\qquad\qquad + \lambda \int_E \{ a\varphi(x) - f(x) \}^2 \varphi(x) \mu(dx). \end{aligned}$$

Then, for all f and a,

$$M_t = \langle f, X_t^B \rangle + a\Lambda_t^B - \langle f, X_0^B \rangle - a\Lambda_0^B - \int_0^t L^{B(1)}(X_s^B, \Lambda_s^B)(f, a)\, ds \tag{4.6}$$

is a martingale, having previsible increasing process

$$\langle M \rangle_t = \int_0^t Q^{B,(1)}(X_s^B, \Lambda_s^B)(f, a)\, ds. \tag{4.7}$$

Recall from §2 that, for $B \subseteq E$ compact, we denote by $\mathcal{M}_B$ the space of finite signed measures supported on B and we define $L^B : \mathcal{M}_B \times \mathbb{R} \to \mathcal{M}_B \times \mathbb{R}$ by the requirement

$$\langle (f,a), L^B(\mu,\lambda)\rangle =$$
$$\frac{1}{2}\int_{E\times E}\{f(x+y)1_{x+y\in B} - a\varphi(x+y)1_{x+y\notin B} - f(x) - f(y)\}K(x,y)\mu(dx)\mu(dy)$$
$$+ \lambda \int_E \{a\varphi(x) - f(x)\}\varphi(x)\mu(dx)$$

for all f and a.

Fix a measure μ_0 on E with $\langle \varphi, \mu_0\rangle < \infty$. Set

$$\mu_0^B = 1_B \mu_0, \quad \lambda_0^B = \langle \varphi 1_{B^c}, \mu_0\rangle.$$

Recall from §2 that, for each compact set B, the equation

$$(\mu_t^B, \lambda_t^B) = (\mu_0^B, \lambda_0^B) + \int_0^t L^B(\mu_s^B, \lambda_s^B)\, ds \tag{4.8}$$

has a unique solution, which is a continuous map

$$t \mapsto (\mu_t^B, \lambda_t^B) : [0,\infty) \to \mathcal{M}_B^+ \times \mathbb{R}^+.$$

Consider now a sequence of integer-valued measures X_0^n. For each n, denote by $(X_t^n)_{t\geq 0}$ and $(X_t^{B,n}, \Lambda_t^{B,n})_{t\geq 0}$ the stochastic coalescent and the coupled family of approximations constructed above, starting from X_0^n. Set

$$\tilde{X}_t^n = n^{-1} X_{n^{-1}t}^n,$$
$$(\tilde{X}_t^{B,n}, \tilde{\Lambda}_t^{B,n}) = n^{-1}(X_{n^{-1}t}^{B,n}, \Lambda_{n^{-1}t}^{B,n}).$$

We shall need a mild continuity condition on K. Denote by $S(K) \subseteq E \times E$ the set of discontinuity points of K and by μ_0^{*n} the n-th convolution power of μ_0. Our assumption is that

$$(\mu_0^{*n})^{\otimes 2}(S(K)) = 0, \qquad \text{for all } n \geq 1. \tag{4.9}$$

This condition is verified, in particular, when $S(K)$ has Lebesgue measure zero and μ_0 is absolutely continuous.

For the purposes of the next proposition, we also need an analogous condition on the compact set B:

$$\mu_0^{*n}(\partial B) = 0, \qquad \text{for all } n \geq 1. \tag{4.10}$$

This condition is verified, for any given μ_0, for all but countably many closed intervals in E.

Proposition 4.2. *Assume conditions* (2.4), (2.5), (4.9), (4.10). *Suppose that*

$$d(\tilde{X}_0^{B,n}, \mu_0^B) \to 0, \quad |\tilde{\Lambda}_0^{B,n} - \lambda_0^B| \to 0$$

as $n \to \infty$. Then, for all $t \geq 0$

$$\sup_{s \leq t} d(\tilde{X}_s^{B,n}, \mu_s^B) \to 0, \quad \sup_{s \leq t} |\tilde{\Lambda}_s^{B,n} - \lambda_s^B| \to 0$$

in probability as $n \to \infty$.

Proof. Set $\Lambda = \sup_n \langle \varphi, \tilde{X}_0^n \rangle$ and note that $\Lambda < \infty$. By rescaling (4.6) and (4.7), we see that, for all B, all bounded measurable functions f and all $a \in \mathbb{R}$

(4.11)

$$M_t^n = \sqrt{n} \left(\langle f, \tilde{X}_t^{B,n} \rangle + a\tilde{\Lambda}_t^{B,n} - \langle f, \tilde{X}_0^{B,n} \rangle - a\tilde{\Lambda}_0^{B,n} - \int_0^t L^{B,(n)}(\tilde{X}_s^{B,n}, \tilde{\Lambda}_s^{B,n})(f,a)\, ds \right)$$

is a martingale, having previsible increasing process

$$\langle M^n \rangle_t = \int_0^t Q^{B,(n)}(\tilde{X}_s^{B,n}, \tilde{\Lambda}_s^{B,n})(f,a)\, ds \tag{4.12}$$

where

$$L^{B,(n)}(\mu, \lambda) = n^{-2} L^{B,(1)}(n\mu, n\lambda),$$
$$Q^{B,(n)}(\mu, \lambda) = n^{-2} Q^{B,(1)}(n\mu, n\lambda).$$

There is a constant $C < \infty$, depending only on B, Λ and φ such that

$$|L^B(\tilde{X}_t^{B,n}, \tilde{\Lambda}_t^{B,n})(f,a)| \leq C(\|f\| + |a|),$$
$$|(L^B - L^{B,(n)})(\tilde{X}_t^{B,n}, \tilde{\Lambda}_t^{B,n})(f,a)| \leq Cn^{-1}(\|f\| + |a|),$$
$$|Q^{B,(n)}(\tilde{X}_t^{B,n}, \tilde{\Lambda}_t^{B,n})(f,a)| \leq C(\|f\| + |a|)^2.$$

Hence by the same argument as in Theorem 4.1, the laws of the sequence $(\tilde{X}^n, \tilde{\Lambda}^n)$ are tight in $D([0,\infty), \mathcal{M}_B \times \mathbb{R})$. Indeed, similarly, the laws of the sequence $(\tilde{X}^n, \tilde{\Lambda}^n, I^n, J^n)$ are tight in $D([0,\infty), \mathcal{M}_B \times \mathbb{R} \times \mathcal{M}_{B\times B} \times \mathcal{M}_{B\times B})$, where

$$I_t^n(dx,dy) = K(x,y) 1_{x+y\in B} \tilde{X}_t^n(dx) \tilde{X}_t^n(dy),$$
$$J_t^n(dx,dy) = K(x,y) 1_{x+y\notin B} \tilde{X}_t^n(dx) \tilde{X}_t^n(dy).$$

Denote by (X, Λ, I, J) some weak limit point of this sequence, which, by passing to a subsequence and the usual argument of Skorohod, we may regard as a pointwise limit in $D([0,\infty), \mathcal{M}_B \times \mathbb{R} \times \mathcal{M}_{B\times B} \times \mathcal{M}_{B\times B})$. Then there exist bounded measurable functions

$$I, J : \Omega \times [0,\infty) \times B \times B \to [0,\infty)$$

symmetric on $B \times B$, such that

$$I_t(dx,dy) = I(t,x,y) X_t(dx) X_t(dy),$$
$$J_t(dx,dy) = J(t,x,y) X_t(dx) X_t(dy)$$

in $\mathcal{M}_{B\times B}$ and such that

$$I(t,x,y) = K(x,y) 1_{x+y\in B}$$
$$J(t,x,y) = K(x,y) 1_{x+y\notin B}.$$

whenever $(x, y) \notin S(K)$ and $x + y \notin \partial B$. Moreover we can pass to the limit in (4.11) to obtain, for all continuous functions f and all $a \in \mathbb{R}$, for all $t \geq 0$, almost surely

$$\begin{aligned} \langle (f,a),(X_t,\Lambda_t)\rangle &= \langle (f,a),(X_0,\Lambda_0)\rangle \\ &\quad + \frac{1}{2}\int_0^t \int_{E\times E} \{f(x+y) - f(x) - f(y)\} I(s,x,y) X_s(dx)X_s(dy)ds \\ &\quad + \frac{1}{2}\int_0^t \int_{E\times E} \{a\varphi(x+y) - f(x) - f(y)\} J(s,x,y) X_s(dx)X_s(dy)ds \\ &\quad + \int_0^t \Lambda_s \int_E \{a\varphi(x) - f(x)\}\varphi(x)X_s(dx)ds \end{aligned} \tag{4.13}$$

By the remarks following the proof of Proposition 2.2, this equation forces $X_t \otimes X_t$ to be absolutely continuous with respect to

$$\sum_{n=1}^{\infty} (\mu_0^{*n})^{\otimes 2}.$$

Hence by the assumptions (4.9), (4.10), we can replace $I(t, x, y)$ by $K(x, y)1_{x+y\in B}$ and $J(t, x, y)$ by $K(x, y)1_{x+y\notin B}$ in (4.13). But this is now equation (4.8) which has a unique solution $(\mu_t^B, \lambda_t^B)_{t\geq 0}$. We have shown that the unique weak limit point of $(\tilde{X}^n, \tilde{\Lambda}^n)$ in $D([0, \infty), \mathcal{M}_B) \times \mathbb{R}$ is the continuous deterministic path $(\mu_t^B, \lambda_t^B)_{t\geq 0}$, which proves the proposition. □

We consider now the special case where μ_0 is a probability measure on $\mathbb{N} = \{1, 2, \dots\}$. Here we can replace the method of weak convergence in Proposition 4.2 by a more direct approach using differential equations. The benefit in this approach, besides greater transparency, is that we can establish a rate of convergence, which is in principle computable. This would be needed if one wished, in practice, to assess whether Smoluchowski's equation provided a tolerable approximation to the stochastic coalescent. Since the stochastic coalescent already makes a mean-field approximation—it is assumed we neglect spatial variations in the particle mass distribution—we are effectively assuming there is some external spatial mixing. The relevant particle number n is then the number of particles in the largest region which is mixed to equilibrium in unit time.

Proposition 4.3. *Assume conditions* (2.4) *and* (2.5)*. Suppose that μ_0 is supported on $\mathbb{N}$. Let B be a finite subset of $\mathbb{N}$. Then there is a constant $C < \infty$, depending only on K, φ, μ_0, and B such that, for all $n \geq 1$, for $\tilde{X}_t^B = n^{-1}X_{n^{-1}t}^B$ and $\tilde{\Lambda}_t^B = n^{-1}\Lambda_{n^{-1}t}^B$, for all $0 \leq \delta \leq t$, if $\delta_0 = \|\tilde{X}_0^B - \mu_0^B\| + |\tilde{\Lambda}_0^B - \lambda_0^B| \leq 1$ then*

$$\mathbb{P}\left(\sup_{s\leq t}\{\|\tilde{X}_s^B - \mu_s^B\| + |\tilde{\Lambda}_s^B - \lambda_s^B|\} > (\delta_0 + \delta)e^{Ct}\right) \leq Ce^{-n\delta^2/Ct}.$$

Proof. We regard $(\tilde{X}^B_t, \tilde{\Lambda}^B_t)_{t\geq 0}$ as taking values in the finite-dimensional vector space $\mathbb{R}^B \times \mathbb{R}$. Recall from the proof of Proposition 4.2 that

$$M_t = \sqrt{n}\{(\tilde{X}^B_t, \tilde{\Lambda}^B_t) - (\tilde{X}^B_0, \tilde{\Lambda}^B_0) - \int_0^t L^{B,(n)}(\tilde{X}^B_s, \tilde{\Lambda}^B_s)\, ds\} \tag{4.14}$$

is a martingale, having previsible increasing process

$$\langle M\rangle_t = \int_0^t Q^{B,(n)}(\tilde{X}^B_s, \tilde{\Lambda}^B_s)\, ds.$$

We recall a form of the exponential martingale inequality for martingales M whose jumps are bounded uniformly by $A \in [0,\infty)$ and which have a continuous previsible increasing process $\langle M\rangle$: for all $\theta \geq 0$ we have

$$\mathbb{P}(\sup_t M_t \geq \delta \text{ and } \langle M\rangle_\infty \leq \varepsilon) \leq \exp\{-\theta\delta + \frac{1}{2}\theta^2 e^{\theta A}\varepsilon\}.$$

Let $f : B \to \mathbb{R}$ and $a \in \mathbb{R}$ be given, with $\|f\| + |a| \leq 1$. Consider the martingale

$$M^{f,a}_t = \langle (f,a), M_t\rangle.$$

The jumps of $M^{f,a}$ are bounded uniformly by $Cn^{-\frac{1}{2}}$ for some $C < \infty$, depending on φ and B. The process $\langle M^{f,a}\rangle$ is continuous and satisfies

$$\langle M^{f,a}\rangle_t \leq Ct$$

for some $C < \infty$, depending on φ, B and μ_0. So, by the exponential martingale inequality

$$\mathbb{P}\big(\sup_{s\leq t} M^{f,a}_s \geq \delta\big) \leq \exp\{-\theta\delta + \frac{1}{2}C\theta^2 t e^{C\theta/\sqrt{n}}\}.$$

Assume that $\delta \leq \sqrt{n}t$ and take $\theta = \delta/(3Ct)$. Then $C\theta/\sqrt{n} \leq 1/3$, so $e^{C\theta/\sqrt{n}} \leq 3/2$ and so

$$\mathbb{P}\big(\sup_{s\leq t} M^{f,a}_s \geq \delta\big) \leq e^{-\delta^2/4Ct}.$$

Since B is finite, we deduce that for some $C < \infty$, depending on φ, B and μ_0

$$\mathbb{P}\big(\sup_{s\leq t} \|M_s\| \geq \delta\big) \leq Ce^{-\delta^2/Ct}$$

whenever $\delta \leq \sqrt{n}t$. Hence

$$\mathbb{P}\big(\sup_{s\leq t} \|n^{-\frac{1}{2}} M_s\| \geq \delta\big) \leq Ce^{-n\delta^2/Ct}$$

whenever $\delta \leq t$. Note also the estimate

$$\|(L^B - L^{B,(n)})(\tilde{X}^B_t, \tilde{\Lambda}^B_t)\| \leq Cn^{-1}, \quad \text{for all } t \geq 0,$$

for some $C < \infty$, depending on φ and B. Set $Y_t = (\tilde{X}_t^B, \tilde{\Lambda}_t^B) - (\mu_t^B, \lambda_t^B)$ and subtract the equations (4.14), (2.6) to obtain

$$Y_t = R_t + \int_0^t L_s^B(Y_s)\,ds$$

where

$$R_t = Y_0 + n^{-\frac{1}{2}} M_t + \int_0^t (L^B - L^{B,(n)})(\tilde{X}_s^B, \tilde{\Lambda}_s^B)\,ds$$

and where

$$\begin{aligned}\langle (f,a), L_t^B(\mu,\lambda)\rangle = \frac{1}{2}\int_{E\times E} &\{f(x+y)1_{x+y\in B} + a\varphi(x+y)1_{x+y\notin B} - f(x) - f(y)\} \\ &\times K(x,y)(\tilde{X}_t^B + \mu_t^B)(dx)\mu(dy) \\ &+ (\tilde{\Lambda}_t^B + \lambda_t^B)\int_E (a\varphi(x) - f(x))\varphi(x)\mu(dx) \\ &+ \lambda \int_E (a\varphi(x) - f(x))\varphi(x)(\tilde{X}_t^B + \mu_t^B)(dx).\end{aligned}$$

Note the estimate

$$\|L_t^B(\mu,\lambda)\| \leq C\|(\mu,\lambda)\|/2$$

where $C < \infty$ depends on φ, B and μ_0. Set $g(t) = \sup_{s\leq t}\|Y_s\|$ and $r(t) = \sup_{s\leq t}\|R_s\|$. Then

$$g(t) \leq r(t) + \frac{1}{2}C\int_0^t g(s)\,ds$$

so $g(t) \leq r(t)e^{Ct/2}$. Now, for $\delta \leq t$, we have

$$\mathbb{P}(r(t) \geq g(0) + \delta/2 + Ct/n) \leq Ce^{-n\delta^2/Ct}.$$

We may assume that $C \geq 4$. If $\delta^2 \leq Ct/n$ we have nothing to prove. Otherwise

$$Ct/n < \delta^2 \leq \tfrac{1}{2}\delta e^{Ct/2}$$

so

$$\mathbb{P}(g(t) \geq (\delta_0 + \delta)e^{Ct}) \leq \mathbb{P}(r(t) \geq g(0) + \delta/2 + Ct/n) \leq Ce^{-n\delta^2/Ct}. \qquad \square$$

Here is the main result of this section.

Theorem 4.4. *Let $K : E \times E \to [0,\infty)$ be a symmetric measurable function and let μ_0 be a measure on E. Assume that*

$$(\mu_0^{*n})^{\otimes 2}(S(K)) = 0 \quad \textit{for all } n \geq 1$$

where $S(K)$ denotes the discontinuity set of K. Assume also that, for some continuous sublinear function $\varphi : E \to (0,\infty)$,

$$K(x,y) \leq \varphi(x)\varphi(y) \quad \textit{for all } x,y \in E$$

and that $\langle \varphi, \mu_0 \rangle < \infty$ *and* $\langle \varphi^2, \mu_0 \rangle < \infty$. *Denote by* $(\mu_t)_{t<T}$ *the maximal strong solution provided by Theorem 2.1. Let* $(X_t^n)_{t\geq 0}$ *be a sequence of stochastic coalescents, with coagulation kernel* K. *Set* $\tilde{X}_t^n = n^{-1} X_{n^{-1}t}^n$ *and suppose that*

$$d(\varphi \tilde{X}_0^n, \varphi \mu_0) \to 0$$

as $n \to \infty$. *Then, for all* $t < T$

$$\sup_{s \leq t} d(\varphi \tilde{X}_s^n, \varphi \mu_s) \to 0$$

in probability, as $n \to \infty$.

Moreover, if μ_0 *is supported on* $\mathbb{N}$, *then for all* $t < T$ *and all* $\delta > 0$ *there are constants* $\delta_0 > 0$ *and* $C < \infty$, *depending only on* K, μ_0, φ, t *and* δ, *such that, for all* n,

$$\|\varphi(\tilde{X}_0^n - \mu_0)\| \leq \delta_0$$

implies

$$\mathbb{P}\big(\sup_{s \leq t} \|\varphi(\tilde{X}_s^n - \mu_s)\| > \delta\big) \leq e^{-n/C}.$$

Proof. Fix $\delta > 0$ and $t < T$. Since $(\mu_t)_{t<T}$ is strong, we can find a compact set B satisfying (4.10) and such that $\lambda_t^B < \delta/2$. Now

$$d(\varphi \tilde{X}_0^n, \varphi \mu_0) \to 0$$

so

$$d(\tilde{X}_0^{B,n}, \mu_0^B) \to 0, \quad |\tilde{\Lambda}_0^{B,n} - \lambda_0^B| \to 0.$$

Hence, by Proposition 4.3

$$\sup_{s \leq t} d(\tilde{X}_s^{B,n}, \mu_s^B) \to 0, \quad \sup_{s \leq t} |\tilde{\Lambda}_s^{B,n} - \lambda_s^B| \to 0$$

in probability as $n \to \infty$. Since $\{\mu_s^B : s \leq t\}$ is compact, we also have

$$\sup_{s \leq t} d(\varphi \tilde{X}_s^{B,n}, \varphi \mu_s^B) \to 0$$

in probability as $n \to \infty$. By Proposition 2.5 and by (4.5), for $s \leq t$

$$\begin{aligned}
\|\varphi(\mu_s - \mu_s^B)\| &= \langle \varphi, \mu_s - \mu_s^B \rangle \leq \lambda_s^B \leq \lambda_t^B < \delta/2 \\
\|\varphi(\tilde{X}_s^n - \tilde{X}_s^{B,n})\| &= \langle \varphi, \tilde{X}_s^n - \tilde{X}_s^{B,n} \rangle \leq \tilde{\Lambda}_s^{B,n} \leq \tilde{\Lambda}_t^{B,n} \\
&\leq \lambda_t^B + |\tilde{\Lambda}_t^{B,n} - \lambda_t^B| \\
&\leq \delta/2 + |\tilde{\Lambda}_t^{B,n} - \lambda_t^B|.
\end{aligned}$$

Now

$$\begin{aligned}
d(\varphi \tilde{X}_s^n, \varphi \mu_s) &\leq \|\varphi(\tilde{X}_s^n - \tilde{X}_s^{B,n})\| + d(\varphi \tilde{X}_s^{B,n}, \varphi \mu_s^B) + \|\varphi(\mu_s - \mu_s^B)\| \\
&\leq \delta + d(\varphi \tilde{X}_s^{B,n}, \varphi \mu_s^B) + |\tilde{\Lambda}_t^{B,n} - \lambda_t^B|
\end{aligned}$$

so

$$\mathbb{P}(\sup_{s\le t} d(\varphi\tilde{X}^n_s, \varphi\mu_s) > \delta) \to 0$$

as $n \to \infty$, as required.

In the case where μ is supported on $\mathbb{N}$, we can argue similarly, replacing the weak metric d by the total variation norm and replacing Proposition 4.3 by Proposition 4.4, to arrive at the desired conclusion. □

Corollary 4.5. *Let K, μ_0, and φ be as in Theorem 4.4. Assume in addition that μ_0 is a probability measure and that $\tilde{X}^n_0$ is the empirical distribution of a sample of size n from μ_0. Then, for all $t < T$*

$$\sup_{s\le t} d(\varphi\tilde{X}^n_s, \varphi\mu_s) \to 0$$

in probability, as $n \to \infty$. Moreover, if μ_0 is supported on $\mathbb{N}$, and if $\langle e^{\alpha\varphi}, \mu_0\rangle < \infty$ for some $\alpha > 0$, then, for all $t < T$ and all $\delta > 0$, there is a constant $C < \infty$, depending only on K, μ_0, φ, t and δ, such that for all n

$$\mathbb{P}(\sup_{s\le t} \|\varphi(\tilde{X}^n_s - \mu_s)\| > \delta) \le e^{-n/C}.$$

Proof. For general μ_0, it suffices to note that

$$d(\varphi\tilde{X}^n_0, \varphi\mu_0) \to 0$$

almost surely as $n \to \infty$, by the strong law of large numbers, and to apply Theorem 4.4.

Suppose now that μ_0 is supported on $\mathbb{N}$. We have

$$\|\varphi(\tilde{X}^n_0 - \mu_0)\| \le 2\langle\varphi 1_{(N,\infty)}, \mu_0\rangle + |\langle\varphi 1_{(N,\infty)}, \tilde{X}^n_0 - \mu_0\rangle| + \langle\varphi 1_{(0,N]}, |\tilde{X}^n_0 - \mu_0|\rangle = I_1 + I_2 + I_3.$$

We can choose N so that $I_1 \le \delta_0/2$. Then by standard exponential estimates we can find $C < \infty$, depending on μ_0, φ, N and δ_0, such that

$$\mathbb{P}(I_2 + I_3 > \delta_0/2) \le e^{-n/C}.$$

On combining this estimate with that found in Theorem 4.4, we deduce

$$\mathbb{P}(\sup_{s\le t} \|\varphi(\tilde{X}^n_s - \mu_s)\| > \delta) \le 2e^{-n/C}$$

as required. □

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