ORNSTEIN–UHLENBECK PROCESSES INDEXED BY THE CIRCLE

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Abstract. We consider the class of stationary, continuous, zero-mean Gaussian processes, indexed by the circle, satisfying a two-point Markov property and taking values in a vector bundle over the circle with given holonomy. We establish, subject to certain additional symmetry properties, a classification of all such processes.

1. Introduction

The problem considered in this paper arises from a desire to understand what are the simplest and most natural evolutions of loops in a Riemannian manifold M. A loop is a continuous map $x : S \to M$, where $S = \mathbb{R}/\mathbb{Z}$ is the circle. In order for a loop to move, one must specify a direction, which is a field of tangent vectors $v(t) \in T_{x(t)}M$, and if the loop is not to break apart, $(v(t) : t \in S)$ should be continuous. In the deterministic case one might specify that each point should follow a geodesic, but how is the initial direction to be chosen? If $v(0) \in T_{x(0)}M$ is given, there is a no canonical way to lift to a continuous field of vectors $(v(t) : t \in S)$.

Our interest is, however, in the random case. We would like to construct a 'Brownian motion of loops', such that, in particular, each point on the loop follows a Brownian motion in M. Let us assume that our loops are sufficiently regular that there is a well-defined notion of parallel translation—smooth or semimartingale loops would do. Then the tangent space over the loop has the structure of a Euclidean vector bundle over S with connection. We choose to look for a vector field along each loop determined by this structure alone. Whilst one might reasonably use more of the local differential structure of the manifold along the loop, for example, curvature, our choice is minimal and motivated therefore by economy. Considerations of continuity and infinite-divisibility suggest a Gaussian distribution for the vector field. Further, we wish to impose maximal symmetry, so that, in particular, the distribution is invariant under rotation of loops. There remain a great many choices, so we impose in addition a two-point Markov property: this leads to a simple stochastic structure and may be considered as a type of locality condition. At this point, in contrast to the deterministic case, where there was no canonical vector field, we find, associated to each invariant subspace of the holonomy of the loop, a canonical two-parameter family of vector fields. The parameters are associated to scale and correlation. This classification theorem is our main result, Theorem 3.1. It is followed by a brief investigation of the properties of these canonical vector fields, in the final section.

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2. Vector bundles over the circle

In this section we establish some basic notation and terminology. Let $E = (E_t : t \in S)$ be a Euclidean vector bundle over the circle with connection, having holonomy τ. By this we mean that each E_t is equipped with an inner product and, for each pair $t, t' \in S$, there is an isometry

$$\tau^+_{t't} : E_t \to E_{t'}.$$

These isometries satisfy $\tau^+_{tt} = I$ and, for $t < t' < t'' < t$, the relation

$$\tau^+_{t''t} = \tau^+_{t''t'}\tau^+_{t't}.$$

Here and throughout, we write $t < t' < t'' < t$ or simply $t < t' < t''$ to mean that one passes t, t', t'' in that order as one moves in the positive sense, that is anticlockwise, around the circle. The holonomy τ is the isometry of E_0 given, for all $t \neq 0$, by

$$\tau = \tau^+_{0t}\tau^+_{t0}.$$

The isometries $\tau^+_{t't}$ define parallel translations anticlockwise around S. The clockwise parallel translations are denoted

$$\tau^-_{tt'} = (\tau^+_{t't})^{-1} : E_{t'} \to E_t.$$

An isomorphism of two such bundles $E, \tilde{E}$ is specified by an isometry $\phi : E_0 \to \tilde{E}_0$ such that

$$\tilde{\tau} = \phi\tau\phi^{-1}.$$

In the case $\tilde{E}_t = E_{s+t}$ there is an isomorphism given by $\phi = \tau^+_{0s}$. Given a connected Euclidean vector bundle E, its isomorphism class is determined uniquely by integers $k \geq 1$, $n_1, \dots, n_k \geq 1$ and distinct angles of rotation $\theta_1, \dots, \theta_k \in (-\pi, \pi]$, for which there exists an orthogonal decomposition

$$E_0 = E_0^1 \oplus \cdots \oplus E_0^k$$

such that

$$\tau = \tau_1 \otimes \cdots \otimes \tau_k,$$

where E_0^i has dimension n_i, with n_i even unless $\theta_i = 0$ or $\theta_i = \pi$, and where $\tau_i = \tau(n_i, \theta_i)$ acts on E_0^i as a rotation through θ_i. Thus

$$\begin{aligned}
\tau(n, 0) &= I, \\
\tau(n, \pi) &= -I, \\
\tau(2n, \theta) &= I\cos\theta + J\sin\theta, \qquad 0 < |\theta| < \pi,
\end{aligned}$$

where J is an isometry with $J^2 = -I$. The bundle then decomposes as

$$E = E^1 \oplus \cdots \oplus E^k$$

where

$$E_t^i = \tau^+_{t0}E_0^i.$$

Lemma 2.1. *Suppose $\gamma : E_0 \to E_0$ is a linear map, which commutes with every isometry of E_0 commuting with τ. Then γ respects the decomposition*

$$E_0 = E_0^1 \oplus \cdots \oplus E_0^k$$

and is given on E_0^i, for some $\lambda_i, \mu_i \in \mathbb{R}$, by

$$\gamma = \lambda_i I + \mu_i \tau_i.$$

Proof. For any orthogonal decomposition $E_0 = E_0^- \oplus E_0^+$ respected by τ, τ commutes with the isometry which acts as ± 1 on $E_0^\pm$, so must γ, and hence γ must also respect the decomposition. Thus we reduce to the following two special cases. When $\tau = I$, we must have $\gamma = \lambda I$ for some $\lambda \in \mathbb{R}$. When $\tau = J$ with $J^2 = -I$, we must have $\gamma = \lambda I + \mu J$ for some $\lambda, \mu \in \mathbb{R}$. □

3. Ornstein–Uhlenbeck processes

Let E be a Euclidean vector bundle over S with connection. Given a continuous section $x : S \to E$, we denote by $\tilde{x} : \mathbb{R} \to E_0$ the continuous lifting given by

$$\tilde{x}_{n+t} = \tau^{-n} \tau_{0t}^{-} x_t, \qquad n \in \mathbb{Z}, \quad 0 \le t < 1.$$

Note, in particular, that $\tilde{x}_1 = \tau^{-1} \tilde{x}_0$.

We now define a number of properties that a continuous random section

$$X : \Omega \times S \to E$$

might have. Although these will be expressed in terms of $\tilde{X}$, it is easy to check that the point 0 plays no special role. We say $(X_t : t \in S)$ is *Gaussian* if $(\tilde{X}_t : t \in \mathbb{R})$ is Gaussian. We say $(X_t : t \in S)$ is *stationary* if $(\tilde{X}_{s+t} : t \in \mathbb{R})$ has the same distribution as $(\tilde{X}_t : t \in \mathbb{R})$, for all $s \in \mathbb{R}$. We say $(X_t : t \in S)$ is *rotation invariant* if $(\phi \tilde{X}_t : t \in \mathbb{R})$ has the same distribution as $(\tilde{X}_t : t \in \mathbb{R})$, for all isometries ϕ commuting with τ. We say $(X_t : t \in S)$ is *Markovian* if, for all $s, t \in S$, conditional on X_s and X_t, $(X_r : s < r < t)$ and $(X_r : t < r < s)$ are independent. We say $(X_t : t \in S)$ is *locally reversible* if, for all $s, t \in \mathbb{R}$ with $0 < t - s < 1$, for all $x, y \in E_0$, the distribution of $(\tilde{X}_{s+r} : 0 < r < t - s)$ given $\tilde{X}_s = x, \tilde{X}_t = y$ is the same as the distribution of $(\tilde{X}_{t-r} : 0 < r < t - s)$ given $\tilde{X}_t = x, \tilde{X}_s = y$. We say $(X_t : t \in S)$ is *reversible* if $(\tilde{X}_{-t} : t \in \mathbb{R})$ has the same distribution as $(\tilde{X}_t : t \in \mathbb{R})$. Note that, given stationarity, reversibility implies local reversibility; but, since

$$\tilde{X}_1 = \tau^{-1} \tilde{X}_0 = \tau^{-2} \tilde{X}_{-1},$$

reversibility must in general fail, unless $\tau^2 = I$.

We say that a continuous random section X of E is an *Ornstein–Uhlenbeck process* if it is stationary, centred, Gaussian, rotation invariant, Markovian and local reversible. We write $X \sim \mathrm{OU}(E)$ for short.

The distribution of any continuous stationary centred Gaussian section $(X_t : t \in S)$ in E is determined by its *covariance*

$$c(t) = \mathbb{E}\tilde{X}_0 \otimes \tilde{X}_t^*, \qquad 0 \le t \le 1$$

which is a continuous map $c : [0,1] \to E_0 \otimes E_0^*$ with $c(1) = c(0)\tau$.

For τ an isometry of E_0 and for $\lambda > 0, 0 < \mu < \pi$ and $0 \le t \le 1$, define

$$\begin{aligned} c_\lambda(t,\tau) &= \frac{\tau \sinh \lambda t + I \sinh \lambda(1-t)}{\sinh \lambda}, \\ c_0(t,\tau) &= \tau t + I(1-t), \\ c_{i\mu}(t,\tau) &= \frac{\tau \sin \mu t + I \sin \mu(1-t)}{\sin \mu}, \\ c_{i\pi}(t,-I) &= I \cos \pi t. \end{aligned}$$

Observe that these functions may be regarded as a single family, parametrized by λ, where $-\pi^2 < \lambda^2 < \infty$, with continuous extension to $\lambda = i\pi$ in the case $\tau = -I$. The following result is due, in the case $n = 1$ and $\tau = 1$, to Klein and Landau [KL], for whom it is a special case of a general classification theorem for periodic Gaussian processes having Osterwalder–Schrader positivity. Their method relies on a representation theorem found in Krein[K], and is quite different to ours.

Theorem 3.1. *For $\tau = \tau(n,\theta)$, $0 \le \sigma^2 < \infty$ and $-\theta^2 \le \lambda^2 < \infty$, there exists an* $\mathrm{OU}(E)$ *process with covariance $\sigma^2 c_\lambda(\cdot,\tau)$, and we obtain all* $\mathrm{OU}(E)$ *processes in this way.*

In general, if τ has decomposition

$$\tau = \tau_1 \otimes \cdots \otimes \tau_k$$

where $\tau_i = \tau(n_i,\theta_i)$ with all θ_i distinct, and if $X^i \sim \mathrm{OU}(E^i)$ with $X^1,\dots,X^k$ independent, then

$$X = X^1 \oplus \cdots \oplus X^k$$

is an $\mathrm{OU}(E)$ *process, and we obtain all* $\mathrm{OU}(E)$ *processes in this way.*

Proof. We deal first with existence. Let $\tau = \tau(n,\theta)$ and fix $0 < \mu < |\theta|$. It suffices to treat the case $\sigma = 1$. Set

$$c(t) = c_{i\mu}(t,\tau) = \frac{\tau \sin \mu t + I \sin \mu(1-t)}{\sin \mu}, \qquad 0 \le t \le 1.$$

Note the identity for $x + y \ne m\pi$

$$\sin(x+z) = \frac{\sin x}{\sin(x+y)} \sin(x+y+z) + \frac{\sin y}{\sin(x+y)} \sin z,$$

from which it follows that, for $0 \le s < t < u \le 1$,

$$c(u-s) = \frac{\sin \mu(t-s)}{\sin \mu t} c(u) + \frac{\sin \mu s}{\sin \mu t} c(u-t).$$

Let $\tilde{X}_0$ be a centred Gaussian random variable in E_0, of variance I. For $k \in \mathbb{Z}$, set $\tilde{X}_k = \tau^{-k}\tilde{X}_0$. Suppose inductively, for $m = 0, 1, 2, \dots$, that we can construct a centred Gaussian process $(\tilde{X}_t : t \in 2^{-m}\mathbb{Z})$ such that, for all $s, t \in 2^{-m}\mathbb{Z}$ with $0 \le t \le 1$, we have

$$\mathbb{E}\tilde{X}_s \otimes \tilde{X}^*_{s+t} = c(t).$$

Set $h = 2^{-m-1}$ and compute

$$\begin{aligned} \operatorname{var}\left(\frac{\sin \mu h}{\sin 2\mu h}(\tilde{X}_0 + \tilde{X}_{2h})\right) &= \left(\frac{\sin \mu h}{\sin 2\mu h}\right)^2 \left(2I + c(2h) + c(2h)^*\right) \\ &= I\left(1 - (\cos\mu - \cos\theta)\frac{\tan \mu h}{\sin \mu}\right). \end{aligned}$$

Since $\mu < |\theta|$, we have $\cos\mu > \cos\theta$, so there is an independent centred Gaussian random variable Y such that $\tilde{X}_h$ has variance I, where

$$\tilde{X}_h = \frac{\sin \mu h}{\sin 2\mu h}(\tilde{X}_0 + \tilde{X}_{2h}) + Y.$$

We repeat this construction, independently, 2^m times to obtain $(\tilde{X}_t : t \in 2^{-m-1}\mathbb{Z} \cap [0, 1])$, then set

$$\tilde{X}_{n+t} = \tau^{-n}\tilde{X}_t, \qquad n \in \mathbb{Z}, t \in 2^{-m-1}\mathbb{Z} \cap [0, 1].$$

Now, for $u \in 2^{-m}\mathbb{Z} \cap (0, 1]$,

$$\mathbb{E}\tilde{X}_h \otimes \tilde{X}^*_u = \frac{\sin \mu h}{\sin 2\mu h}\big(c(u) + c(u - 2h)\big) = c(u - h)$$

and, for $u \in 2^{-m}\mathbb{Z} \cap (0, 1)$,

$$\mathbb{E}\tilde{X}_h \otimes \tilde{X}^*_{u+h} = \left(\frac{\sin \mu h}{\sin 2\mu h}\right)^2 \big(c(u - 2h) + 2c(u) + c(u + h)\big) = c(u).$$

Hence, for $s, t \in 2^{-m-1}\mathbb{Z}$ with $0 \le t \le 1$, we have

$$\mathbb{E}\tilde{X}_s \otimes \tilde{X}^*_{s+t} = c(t).$$

This completes the inductive step. Since c is Lipschitz continuous, the usual application of Kolmogorov's criterion now gives us a continuous centred Gaussian process $(\tilde{X}_t : t \in \mathbb{R})$ with

$$\mathbb{E}\tilde{X}_s \otimes \tilde{X}^*_{s+t} = c(t)$$

for all $s \in \mathbb{R}$ and $t \in [0, 1]$. Hence we obtain a continuous stationary centred Gaussian section of E on setting

$$X_t = \tau^+_{t0}\tilde{X}_t.$$

We now have to check that $(X_t : t \in S)$ is rotation invariant, Markovian and locally reversible.

The first is easy. For if ϕ is an isometry of E_0, commuting with τ, then, for all $s \in \mathbb{R}$ and $t \in [0,1]$,

$$\mathbb{E}\phi\tilde{X}_s \otimes (\phi\tilde{X}_{s+t})^* = \phi c(t)\phi^{-1} = c(t).$$

So $(X_t : t \in S)$ is rotation invariant. Next, for $0 \le s < t \le u \le 1$, we can write

$$\tilde{X}_s = \frac{\sin\mu(t-s)}{\sin\mu t}\,\tilde{X}_0 + \frac{\sin\mu s}{\sin\mu t}\,\tilde{X}_t + Y_s.$$

Then we have

$$\mathbb{E}Y_s \otimes \tilde{X}_u^* = c(u-s) - \frac{\sin\mu(t-s)}{\sin\mu t}\,c(u) - \frac{\sin\mu s}{\sin\mu t}\,c(u-t) = 0$$

so Y_s and $\tilde{X}_u$ are independent. This shows that $(X_t : t \in S)$ is Markovian. To establish local reversibility, it suffices to show that $(Y_s : 0 \le s \le t)$ and $(Y_{t-s} : 0 \le s \le t)$ have the same distribution. Then, by the Markov property, it suffices to show that the centred Gaussian random variables Y_s and Y_{t-s} have the same distribution for each $s \in [0,t]$. But

$$\begin{aligned} I &= \operatorname{var}\tilde{X}_s \\ &= \frac{I\sin^2\mu(t-s) + \big(c(t)+c(t)^*\big)\sin\mu(t-s)\sin\mu s + I\sin^2\mu s}{\sin^2\mu t} + \operatorname{var}Y_s \end{aligned}$$

so $\operatorname{var}Y_s = \operatorname{var}Y_{t-s}$ as required.

We have shown that $(X_t : t \in S) \sim \mathrm{OU}(E)$. The argument we used, with obvious modifications, shows also that there exist continuous stationary centred Gaussian sections with covariances $c_\lambda(\cdot,\tau)$ for $\lambda = i|\theta|$ and $\lambda \ge 0$, and moreover these are all $\mathrm{OU}(E)$. We note that, for $-\theta^2 \le \lambda^2 < \infty$, there is a unique function $\gamma = \gamma_\lambda : [0,1/2) \to (0,\infty)$ such that, for all $t \in [0,1/2)$, we have

$$\tilde{X}_t = \gamma(t)(\tilde{X}_0 + \tilde{X}_{2t}) + Y$$

with Y independent of $\tilde{X}_0$ and $\tilde{X}_{2t}$. Moreover we have

$$\begin{aligned} \gamma_\lambda(t) &= \tfrac{1}{2}\operatorname{sech}\lambda t, & \lambda &\ge 0 \\ \gamma_{i\mu}(t) &= \tfrac{1}{2}\sec\mu t, & 0 &\le \mu \le |\theta|. \end{aligned}$$

For $\theta \ne \pi$ and $-\theta^2 \le \lambda^2 < \infty$, each of these functions γ_λ extends continuously to $[0,1/2]$, and $\gamma_\lambda(1/2)$ can take any value in $(0,(1/2)\sec(\theta/2)]$. For $\theta = \pi$ and $-\theta^2 \le \lambda^2 < \infty$, $\gamma_\lambda(1/4)$ can take any value in $(0,1/\sqrt{2}]$.

In the general case we have

$$E_0 = E_0^1 \oplus \cdots \oplus E_0^k$$

and

$$\tau = \tau_1 \otimes \cdots \otimes \tau_k$$

with $\tau_i = \tau(n_i, \theta_i)$ for all i and all θ_i distinct. Then the only isometries of E_0 commuting with τ are those which preserve E_0^i and commute with τ_i for all i. Hence, it is easy to check that, if $X^i \sim \mathrm{OU}(E^i)$ for all i, and $X^1, \dots, X^k$ are independent, then $X = X^1 \oplus \cdots \oplus X^k \sim \mathrm{OU}(E)$.

We turn to the problem of uniqueness. Suppose that $(X_t : t \in S) \sim \mathrm{OU}(E)$. Since $(X_t : t \in S)$ is rotation invariant, we have

$$c(t) = \phi c(t) \phi^{-1}$$

for all isometries ϕ commuting with τ. By Lemma 2.1, $c(t)$ respects the decomposition $E_0^1 \oplus \cdots \oplus E_0^k$ and is given on E_0^i, for some $a_i(t), b_i(t) \in \mathbb{R}$, by

$$c(t) = a_i(t) I + b_i(t) \tau_i.$$

The components $X^1, \dots, X^k$ are therefore independent and, it is easy to check, $X^i \sim \mathrm{OU}(E^i)$ for all i. Thus we are reduced to the case where $\tau = \tau(n, \theta)$ for some $\theta \in (-\pi, \pi]$ and

$$c(t) = a(t) I + b(t) \tau$$

for some $a(t), b(t) \in \mathbb{R}$. Since $c(0)$ is non-negative definite, either X is identically zero, or, after a suitable rescaling, we can assume that $c(0) = I$.

Since $(X_t : t \in S)$ is Gaussian, for each $t \in [0, 1/2]$, there are linear maps $\gamma_-(t), \gamma_+(t)$ on E_0 such that

$$\tilde{X}_0 = \gamma_-(t) \tilde{X}_{-t} + \gamma_+(t) \tilde{X}_t + Y \tag{A}$$

where Y is independent of $\tilde{X}_{-t}, \tilde{X}_t$. By local reversibility we must have $\gamma_-(t) = \gamma_+(t) = \gamma(t)$ say. Moreover

$$c(t) + c(t)^* = \mathbb{E} \tilde{X}_0 \otimes (\tilde{X}_{-t} + \tilde{X}_t)^* = \gamma(t) \mathrm{var}\, (\tilde{X}_{-t} + \tilde{X}_t).$$

If $\tilde{X}_{-t} + \tilde{X}_t = 0$, we can take $\gamma(t) = 0$. If not, then $\mathrm{var}\, (\tilde{X}_{-t} + X_t) = 2I + c(2t) + c(2t)^*$ is invertible. So, in any case, $\gamma(t)$ is a scalar. By the Markov property, stationarity and continuity, the law of $(X_t : t \in S)$ is determined once we know $\gamma(2^{-n}), n = 1, 2, \dots$. We begin by obtaining an upper bound on $\gamma(1/2)$. Since

$$\tilde{X}_{1/2} = \gamma(1/2)(\tilde{X}_0 + \tilde{X}_1) + Y$$

with Y independent of $\tilde{X}_0$, we have

$$I = \mathrm{var}\, \tilde{X}_{1/2} \geq 2(1 + \cos\theta) \gamma(1/2)^2 I$$

so $\gamma(1/2) \leq (1/2) \sec(\theta/2)$. When $\theta = \pi$, this is uninformative, so we seek instead a bound on $\gamma(1/4)$.

We have

$$\tilde{X}_{1/4} = \gamma(1/4)(\tilde{X}_0 + \tilde{X}_{1/2}) + Y$$

with Y independent of $\tilde{X}_0, \tilde{X}_{1/2}$, and moreover, in this case, $\tilde{X}_{1/2}$ is independent of $\tilde{X}_0$. See (ii) below. Hence, for $\theta = \pi$,

$$I = \operatorname{var} \tilde{X}_{1/4} \geq 2\gamma(1/4)^2 I$$

so $\gamma(1/4) \leq 1/\sqrt{2}$.

We claim that $c(t)$ and $\gamma(t)$ have the following properties:

(i) $c(0) = I, c(1) = \tau$;
(ii) $c(t) = \gamma(t)(c(2t) + I)$, $0 \leq t \leq 1/2$;
(iii) $c(2t) = \gamma(t)^2(c(4t) + 2c(2t) + I)$, $0 \leq t \leq 1/4$;
(iv) for $0 < s \leq t \leq 1/2$, if $c(s) = 0$ then $c(t) = 0$.

Property (i) is obvious. To obtain (ii), multiply (A) by X_t^* on the right and take expectations. By stationarity and the Markov property, for $0 \leq t \leq 1/4$

$$X_t = \gamma(t)(X_0 + X_{2t}) + Y',$$
$$X_{-t} = \gamma(t)(X_{-2t} + X_0) + Y''$$

where Y', Y'' are, in particular, independent of X_{2t}. Hence, by substituting in (A) we obtain

$$X_0 = \gamma(t)^2(X_{-2t} + 2X_0 + X_{2t}) + Y''' \tag{B}$$

where Y''' is independent of X_{2t}. To obtain (iii), multiply (B) by X_{2t}^* on the right and take expectations. Property (iv) follows from the Markov property.

We claim, for $\theta \neq \pi$, that the above properties force $\gamma(t) > 0$ for all $t \leq 1/2$. Then, clearly, once $\gamma(1/2) \in (0, (1/2)\sec(\theta/2)]$ is specified, they also determine $\gamma(2^{-n}), n = 2, 3, \ldots$, and hence the law of $(X_t : t \in S)$. Since we have constructed examples for all $\gamma(1/2) \in (0, (1/2)\sec(\theta/2)]$, these must form an exhaustive list. In the case $\theta = \pi$, we claim that the above properties force $\gamma(t) > 0$ for all $t \leq 1/4$. Then, once $\gamma(1/4) \in (0, 1/\sqrt{2}]$ is specified, they also determine $\gamma(2^{-n}), n = 3, 4, \ldots$, and hence, since $\gamma(1/2)$ is unimportant here, the law of $(X_t : t \in S)$. We conclude as before.

It remains to prove the above claims. Suppose that $c(2t) = -I$ for some $t \leq 1/2$; then $c(t) = 0$ by (ii), so $c(1/2) = 0$ by (iv), so, provided $\theta \neq \pi$, $\gamma(1/4) = 0$ by (iii), but then $c(1/4) = 0$ by (ii), so $\gamma(1/8) = 0$ by (iii), and so on. We are forced to conclude that $c(2^{-n}) = 0$ for $n = 1, 2, \ldots$, which contradicts the fact that c is continuous, with $c(0) = I$. Assume that $\theta \neq \pi$. Then $c(2t) \neq -I$ so $c(2t) + I$ is invertible, so $\gamma : [0, 1/2] \to \mathbb{R}$ is continuous. We note from (ii) that $\gamma(t) \to 1/2$ as $t \to 0$. Suppose that $\gamma(t) = 0$ for some $t \leq 1/2$, then $c(t) = 0$ by (ii) and we get a contradiction, as before. Hence $\gamma(t) > 0$ for all t, as claimed.

We turn finally to the case $\theta = \pi$. In this case $c(t)$ is also a scalar and, taking $t = 1/2$ in (ii), we have $c(1/2) = 0$. Suppose $c(t) \leq 0$ for some $t \leq 1/4$, then, by continuity of c, together with (iv), $c(1/4) = 0$. Hence $\gamma(1/8) = 0$ by (iii), $c(1/8) = 0$

by (ii), and so on. We get a contradiction as above. Hence $c(t) > 0$ for all $t \le 1/4$, so $\gamma(t) > 0$ for all $t \le 1/4$ by (ii). □

4. Further properties

We have classified, in terms of their covariance structure, all the Ornstein–Uhlenbeck processes in a Euclidean vector bundle E over S with connection. In particular, we have shown that, if $X \sim \mathrm{OU}(E)$, then X decomposes into independent components $X^i \sim \mathrm{OU}(E^i)$ corresponding to the canonical decomposition of E. So, for convenience and without loss, we shall assume in this section that $\tau = \tau(n, \theta)$ for some $\theta \in (-\pi, \pi]$. Just as Brownian motion and more classical Ornstein–Uhlenbeck processes have many different characterizations, so do the processes considered here. These alternative characterizations form the subject of this section.

First of all, from the proof of Theorem 3.1, we have the following Markovian description. The Ornstein–Uhlenbeck process with covariance $c_\lambda(\cdot, \tau)$, for $-\theta^2 \le \lambda^2 < \infty$, is characterized among stationary continuous Gaussian sections by the properties:

(i) $\operatorname{var}(\tilde{X}_0) = I$,

(ii) for $0 \le s < t \le 1$ we have

$$\tilde{X}_s = \frac{\sinh \lambda(t-s)}{\sinh \lambda t}\, \tilde{X}_0 + \frac{\sinh \lambda s}{\sinh \lambda t}\, \tilde{X}_t + Y$$

for some random variable Y, independent of $(\tilde{X}_u : t \le u \le 1)$. Moreover (ii) may be replaced by the weaker condition:

(ii)′ for $0 < t \le 1/2$ we have

$$\tilde{X}_t = \operatorname{sech} \lambda t\, (\tilde{X}_0 + \tilde{X}_{2t})/2 + Y$$

for some random variable Y, independent of $(\tilde{X}_u : 2t \le u \le 1)$.

Secondly, there is a Hilbert space approach. The space of (covariantly) absolutely continuous sections $h : S \to E$ contains a family of Hilbert spaces, parametrized by λ, where $-\theta^2 < \lambda^2 < \infty$, with norms

$$\|h\|_\lambda^2 = \int_0^1 |\nabla h_t|^2 dt + \lambda^2 \int_0^1 |h_t|^2 dt,$$

where

$$\nabla h_t = \tau_{t0}^+ \frac{d}{dt} (\tau_{0t}^- h_t).$$

The generalized Wiener space construction then gives rise to a continuous centred Gaussian section X characterized by

$$\mathbb{E}\, \langle h, X \rangle_\lambda^2 = \|h\|_\lambda^2.$$

By a standard type of calculation, we can then show that X has covariance given by

$$\mathbb{E}\tilde{X}_0 \otimes \tilde{X}_t^* = \frac{\sinh\lambda}{2\lambda(\cosh\lambda - \cos\theta)} c_\lambda(t,\tau).$$

Hence we obtain in this way, up to a scaling, all $\mathrm{OU}(E)$ processes, except the maximally correlated, smooth process corresponding to $\lambda = i\theta$. Notice that the marginal variance of X depends on the angle of rotation θ, so if this construction is applied in the general case $\tau = \tau_1 \otimes \cdots \otimes \tau_k$, then the marginal variance of X will not be a multiple of the identity. Of course, if this is regarded as a defect, it can be remedied by a suitable reweighting of the Hilbert norms across the decomposition $E = E^1 \oplus \cdots \oplus E^k$. We note that the generalized Wiener space machinery deals quickly with the construction problem, which we considered at greater length by elementary means. It would be interesting to have a direct proof of uniqueness by this method, but it may be that to set up the Hilbert space machinery would require greater *a priori* regularity of X than we have assumed.

In the case where $\lambda \geq 0$, the differential equation

$$\dot{\tilde{h}}_t = \dot{g}_t - \lambda \tilde{h}_t$$

gives rise to an isomorphism of the Hilbert space of X and the classical Cameron–Martin space of Brownian motion. For

$$\|g\|^2 \equiv \int_0^1 |\dot{g}_t|^2 dt = \int_0^1 |\nabla h_t + \lambda h_t|^2 dt = \|h\|_\lambda^2.$$

So we can construct X by solving the linear stochastic differential equation with boundary conditions, driven by Brownian motion

$$d\tilde{X}_t = dB_t - \lambda \tilde{X}_t dt, \quad \tilde{X}_1 = \tau^{-1}\tilde{X}_0.$$

We note that no genuine stochastic integrals are needed for this equation, so the non-adapted boundary condition poses no problem. A general study of some related equations has been made by Ocone and Pardoux [OP].

Explicit solution of the differential equation leads to another representation of X, in terms of an integral kernel with respect to Brownian motion. We shall first present the integral kernel construction, then rederive the differential equation.

Let $(B_t : 0 \leq t < 1)$ be a Brownian motion in E_0 and define a Gaussian process $(\tilde{B}_t : t \in \mathbb{R})$ with stationary increments by $\tilde{B}_0 = 0$ and

$$d\tilde{B}_{n+t} = \tau^{-n} dB_t, \quad n \in \mathbb{Z}, 0 \leq t < 1.$$

Fix $\lambda > 0$ and set

$$\phi(t) = e^{-\lambda(t-1/2)} 1_{0 \leq t < 1},$$

then define

$$\begin{aligned}\tilde{X}_t &= \left(\frac{\lambda}{\sinh\lambda}\right)^{1/2} \int_{\mathbb{R}} \phi(t-r) d\tilde{B}_r \\ &= \left(\frac{\lambda}{\sinh\lambda}\right)^{1/2} \Big\{ \int_0^t e^{-\lambda(t-r-1/2)} dB_r + \tau \int_t^1 e^{-\lambda(t-r+1/2)} dB_r \Big\}. \qquad \text{(C)}\end{aligned}$$

Then $\tilde{X}_{n+t} = \tau^{-n}\tilde{X}_t$ for $n \in \mathbb{Z}$ and $0 \le t < 1$, so we obtain a continuous, stationary, centred, Gaussian, rotation invariant section of E on setting

$$X_t = \tau_{t0}^{+}\tilde{X}_t, \qquad 0 \le t < 1.$$

Moreover, for $0 \le t \le 1$, we have

$$\begin{aligned}
\mathbb{E}\tilde{X}_0 \otimes \tilde{X}_t^* &= \left(\frac{\lambda}{\sinh\lambda}\right)\left\{\tau\int_0^t e^{\lambda(r-1/2)}e^{\lambda(r-t+1/2)}dr + I\int_t^1 e^{\lambda(r-1/2)}e^{\lambda(r-t-1/2)}dr\right\} \\
&= \frac{\tau\sinh\lambda t + I\sinh\lambda(1-t)}{\sinh\lambda} = c_\lambda(\tau,t) \qquad \blacksquare
\end{aligned}$$

and on differentiating (C) we obtain

$$\begin{aligned}
d\tilde{X}_t &= (\lambda/\sinh\lambda)^{1/2}(e^{\lambda/2} - \tau e^{-\lambda/2})dB_t - \lambda\tilde{X}_t dt \\
&= \sigma(\lambda,\theta)dW_t - \lambda\tilde{X}_t dt
\end{aligned}$$

where

$$\sigma^2(\lambda,\theta) = 2\lambda(\cosh\lambda - \cos\theta)/\sinh\lambda$$

and where $(W_t : 0 \le t < 1)$ is a Brownian motion in E_0. So we recover, up to a scaling, the differential equation discussed above.

We can also characterize X by means of a stochastic differential equation, adapted to the filtration

$$\mathcal{F}_t = \sigma\{\tilde{X}_s : 0 \le s \le t\}.$$

Consider for now the case $\lambda > 0$. For $0 \le t < t+h \le 1$ we know that

$$\tilde{X}_{t+h} = \frac{\sinh\lambda h}{\sinh\lambda(1-t)}\,\tilde{X}_1 + \frac{\sinh\lambda(1-t-h)}{\sinh\lambda(1-t)}\,\tilde{X}_t + Y_{t,t+h}$$

where $Y_{t,t+h}$ is a centred Gaussian, independent of $\mathcal{F}_t$. So, as $h \downarrow 0$, uniformly in $t \le 1-\varepsilon$ for each $\varepsilon > 0$, we have

$$\begin{aligned}
\tilde{X}_{t+h} - \tilde{X}_t = &\left(\frac{\lambda h}{\sinh\lambda(1-t)} + \mathrm{o}(h)\right)\tau^{-1}\tilde{X}_0 \\
&- \left(\frac{\lambda h\cosh\lambda(1-t)}{\sinh\lambda(1-t)} + \mathrm{o}(h)\right)\tilde{X}_t + Y_{t,t+h}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\operatorname{var}(Y_{t,t+h}) &= \operatorname{var}(\tilde{X}_{t+h} - \tilde{X}_t) + \mathrm{o}(h) \\
&= -2c'_\lambda(0+,\tau)h + \mathrm{o}(h) \\
&= \sigma^2(\lambda,\theta)hI + \mathrm{o}(h).
\end{aligned}$$

Define $(B_t : 0 \le t < 1)$ by

$$\begin{aligned}
\sigma(\lambda,\theta)B_t &= \tilde{X}_t - \tilde{X}_0 - \int_0^t \lambda\tau^{-1}\tilde{X}_0\,\mathrm{cosech}\,\lambda(1-s)ds + \int_0^t \lambda\tilde{X}_t\coth\lambda(1-s)ds \\
&= \lim_{n\to\infty}\sum_{k=0}^{[nt]-1} Y_{k/n,(k+1)/n}.
\end{aligned}$$

Then B is a standard Brownian motion in E_0, adapted to $(\mathcal{F}_t)_{0\leq t<1}$ and $\tilde{X}$ satisfies the stochastic differential equation

$$d\tilde{X}_t = \sigma(\lambda,\theta)dB_t + \lambda\tau^{-1}\tilde{X}_0 \operatorname{cosech}\lambda(1-t)dt - \lambda\tilde{X}_t \coth\lambda(1-t)dt.$$

This all works, with obvious modifications, for the cases $\lambda = 0$ and $\lambda = i\mu$, $0 < \mu < |\theta|$. We note that the representation of X using a stochastic differential equation with boundary conditions works, by contrast, only for $\lambda \geq 0$.

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