INTEGRATION BY PARTS AND CAMERON–MARTIN FORMULAS FOR THE FREE PATH SPACE OF A COMPACT RIEMANNIAN MANIFOLD

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1. Introduction

Let $(h_s : s \geq 0)$ denote an absolutely continuous function with values in $\mathbb{R}^m$ whose derivative is square-integrable:

$$\int_0^\infty |\dot{h}_s|^2 ds < \infty.$$

The Cameron–Martin formula states that if $(x_s : s \geq 0)$ is a Brownian motion in $\mathbb{R}^m$, starting from 0, then, *provided also* $h_0 = 0$, the law of $(x_s + h_s : s \geq 0)$ is absolutely continuous with respect to that of $(x_s : s \geq 0)$ with density

$$\rho_s = \exp\left\{ \int_0^\infty \langle \dot{h}_s, dx_s \rangle - \tfrac{1}{2} \int_0^\infty |\dot{h}_s|^2 ds \right\}.$$

In fact if one randomizes the starting point x_0 according to Lebesgue measure, then the formula remains valid without the assumption $h_0 = 0$. Thus we obtain a Cameron–Martin formula for the free path space of $\mathbb{R}^m$. For suitable functions F on the path space, the expectation

$$\mathbb{E}\left[F(x + th) \exp\left\{ - \int_0^\infty \langle t\dot{h}_s, dx_s \rangle - \tfrac{1}{2} \int_0^\infty |t\dot{h}_s|^2 ds \right\} \right]$$

does not depend on t. So on differentiating in t at 0 we obtain an integration by parts formula:

$$\mathbb{E}[D_h F(x)] = \mathbb{E}\big[F(x) \int_0^\infty \langle \dot{h}_s, dx_s \rangle \big].$$

This may be regarded as the infinitesimal form of the Cameron–Martin formula.

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In this note we shall discuss Cameron–Martin and integration-by-parts formulas for the free path space of a compact Riemannian manifold. The case of paths with a fixed starting point has already been thoroughly discussed: see [D],[H]. The results we obtain are at a technical level simple corollaries of results in [L2] or [N2]. The emphasis here is rather on the efficient calculation of densities and divergences for flows and vector fields. The integration by parts formula is proved first in §2, by a direct argument based on the methods of [L2]. Then in §3 we use the main result of [N2] to establish independently a corresponding Cameron–Martin formula. From here we can recover the integration by parts formula by differentiating.

Integration by parts formulas of a similar type are proved in [L1],[L2],[LR] by using a mixture of small time asymptotics and developments of Bismut's formula [B],[EL],[N1]. They rely deeply on the identity between the tangent spaces to path space used by Bismut [B] and Jones and Léandre [JL]. Such integration by parts formulas are also known for free twisted loops: see [LR]. In this case we do not yet have a corresponding Cameron–Martin formula.

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2. An integration by parts formula on the free path space

Let Ω denote the set of continuous paths $(x_s : s \geq 0)$ with values in a compact Riemannian manifold M. Let X denote a vector field on Ω, thus $X(x) = \big(X_s(x) : s \geq 0\big)$ where $X_s(x)$ belongs to the tangent space to M at x_s. We shall investigate the relationship between X and the equilibrium Wiener measure on Ω:

$$\mathbb{P}(dx) = \int_M \mathbb{P}^{x_0}(dx)dx_0$$

where $\mathbb{P}^{x_0}$ denotes the law of Brownian motion in M starting from x_0 and dx_0 denotes the normalized Riemannian volume. Let us consider the vector field X given by

$$X_s(x) = \tau_s \sum_{i=1}^m h_s^i X_i(x_0)$$

where, for $i = 1, \dots, m$, X_i is a C^2 vector field over M and τ_s is the parallel transport from x_0 to x_s.

Conditional on x_0, we define a Brownian motion b_s in $T_{x_0}M$ by $b_0 = 0$ and

$$\partial b_s = \tau_s^{-1} \partial x_s$$

where ∂ denotes the Stratonovich differential.

For each $s \geq 0$ denote by $e_s : \Omega \to M$ the evaluation map $e_s(x) = x_s$. We consider the pullback T_s by e_s of the tangent bundle TM equipped with the pullback of the Levi–Civita connection of M. Then formally ∂x_s is a section of T_s, ∂b_s is a section of T_0, τ_s is a section of $T_s \otimes (T_0)^*$, and we have

$$\nabla_X \partial x_s = (\nabla_X \tau_s)\partial b_s + \tau_s \nabla_X \partial b_s.$$

This formula is justified in [L2] at (4.65). We know also by [L2] (see (4.64), (3.87)) that

$$\nabla_X \tau_s = \tau_s \int_0^s \tau_r^{-1} R(\partial x_r, X_r)\tau_r,$$
$$\nabla_X \partial x_s = \tau_s \sum_{i=1}^m \dot h_s^i X_i(x_0)\partial s$$

so

$$\nabla_X \partial b_s = \sum_{i=1}^m \dot h_s^i X_i(x_0)\partial s - \Big(\int_0^s \tau_r^{-1} R(\partial x_r, X_r)\tau_r\Big)\partial b_s.$$

Hence we obtain for the Itô differential

$$\nabla_X db_s = \sum_{i=1}^m \dot h_s^i X_i(x_0)ds - \tfrac12 \tau_s^{-1}\mathrm{Ricci}(X_s)ds - \Big(\int_0^s \tau_r^{-1} R(\partial x_r, X_r)\tau_r\Big)db_s.$$

We compute now the action of X on a test functional

$$F = \sum_n \int_{0<s_1<\cdots<s_n} H(s_1,\dots,s_n;x_0)db_{s_1}\dots db_{s_n}$$

where the sum in n is finite. Here H is a cotensor in $T_{x_0}M$. We have

$$\begin{aligned}\langle dF, X\rangle = &\sum_n \int_{0<s_1<\cdots<s_n} \sum_j H(s_1,\dots,s_n;x_0)db_{s_1}\dots \nabla_X db_{s_j}\dots db_{s_n}\\ &+\sum_n \int_{0<s_1<\cdots<s_n} \nabla_{X_0} H(s_1,\dots,s_n;x_0)db_{s_1}\dots db_{s_n},\end{aligned}$$

so

$$\begin{aligned}\mathbb{E}\langle dF, X\rangle = &\int_M dx_0 \mathbb{E}^{x_0}\Big(\sum_n \int_{0<s_1<\cdots<s_n} H(s_1,\dots,s_n;x_0)db_{s_1}\dots db_{s_{n-1}}(\theta_{s_n} ds_n)\Big)\\ &+\int_M X_0 f(x_0)dx_0\end{aligned}$$

where $f(x_0) = \mathbb{E}^{x_0}(F)$ and

$$\theta_s = \tau_s^{-1}\big(D/\partial s - \tfrac12\mathrm{Ricci}\big)X_s.$$

Here $D/\partial s$ denotes covariant differentiation along x_s. In the first term we used the fact that integrals in db_s vanish under the expectation. In the second we used the fact that the Fock space structure, being derived from the metric, is preserved by the Levi–Civita connection. Let us define

$$\operatorname{div} X(x) = \operatorname{div} X_0(x_0) + \int_0^\infty \langle \Big(\frac{D}{\partial s} - \tfrac12\mathrm{Ricci}\Big) X_s(x), dx_s\rangle.$$

We have shown:

Theorem 2.1. *We have*

$$\mathbb{E}\langle dF, X\rangle = \mathbb{E}(F\operatorname{div} X).$$

3. A Cameron–Martin formula on the free path space

Recall that Ω denotes the set of continuous paths $(x_s : s \geq 0)$ with values in M. Let X denote a vector field on Ω, thus $X(x) = (X_s(x) : s \geq 0)$ with $X_s(x) \in T_{x_s}M$. Our object now is to compute the image of the equilibrium Wiener measure $\mathbb{P}$ under the flow determined by X.

We begin with a rough argument from which some technical points are missing. Later, in order to fill these gaps we shall specialize our choice of vector field X, which may obscure the simplicity of the basic argument. Let us assume that X_s is previsible, and that $DX_s/\partial s$ exists for almost all s, and is square-integrable. Let us assume also that for $\mathbb{P}$-almost all $x_0 \in \Omega$, we can integrate X to a flow in Ω

$$\dot{x}_t = X(x_t). \tag{1}$$

Here we use t to parametrize a family of paths $x_t = (x_{st} : s \geq 0)$. Let us suppose that x_{st} is a two-parameter semimartingale in the sense of [N2], then the two-parameter stochastic calculus provides a means to compute the law of x_t when x_0 has law $\mathbb{P}$.

We may rewrite (1) in differential form

$$\partial_t x_{st} = X_{st}\partial t$$

where $X_{st} = X_s(x_t)$. Recall that we write d_s and ∂_s for the Itô and Stratonovich differentials in s; we also write D_s for the covariant Stratonovich differential corresponding to the Levi–Civita connection. Then

$$D_s\partial_t x_{st} = \left(\frac{D}{\partial s}X_{st}\right)\partial s\partial t.$$

Let us introduce a lift v_{st} of x_{st} to the bundle OM of orthonormal frames in TM, choosing v_{00} arbitrarily and imposing the following horizontality conditions:

$$D_s v_{s0} = 0, \quad D_t v_{st} = 0,$$

which determine v_{st} uniquely, given v_{00}. In addition we introduce two further processes, q_{st} in TM over x_{st}, and b_{st} in $\mathbb{R}^n$, by the equations

$$D_t q_{st} = \left(\frac{D}{\partial s} - \tfrac{1}{2}\mathrm{Ricci}\right)X_{st}\partial t, \quad q_{s0} = 0,$$
$$d_s b_{st} = v_{st}^{-1}(d_s x_{st} - q_{st}ds), \quad b_{0t} = 0.$$

Since x_{s0} is a Brownian motion in M, it follows that b_{s0} is a Brownian motion in $\mathbb{R}^n$. Since our connection is torsion-free,

$$D_s\partial_t x_{st} = D_t\partial_s x_{st},$$

hence
$$\partial_t(\partial_s b_{st} \otimes \partial_s b_{st}) = v_{st}^{-1} D_t(\partial_s x_{st} \otimes \partial_s x_{st}) = 0$$
and so
$$\partial_s b_{st} \otimes \partial b_{st} = \partial_s b_{s0} \otimes \partial_s b_{s0} = \sum_{i=1}^{n} e_i \otimes e_i \partial s,$$
where e_i runs over the standard basis in $\mathbb{R}^n$. We recall the basic identity ([N2], (2.38))
$$D_t \partial_s x_{st} = D_t d_s x_{st} + \tfrac{1}{2} R(\partial_t x_{st}, \partial_s x_{st}) \partial_s x_{st},$$
where R denotes the curvature. But we have identified the quadratic variation in s of x_{st} as the trace, so
$$R(\partial_t x_{st}, \partial_s x_{st}) \partial_s x_{st} = \mathrm{Ricci}(\partial_t x_{st}) \partial s.$$

Hence
$$\begin{aligned} D_t d_s x_{st} &= D_t \partial_s x_{st} - \tfrac{1}{2} \mathrm{Ricci}(\partial_t x_{st}) \partial s \\ &= \left(\frac{D}{\partial s} - \tfrac{1}{2} \mathrm{Ricci} \right) X_{st} \partial s \partial t \\ &= D_t q_{st} \partial s, \end{aligned}$$

and
$$\partial_t d_s b_{st} = v_{st}^{-1}(D_t d_s x_{st} - D_t q_{st} ds) = 0.$$
Therefore $b_{st} = b_{s0}$ for all t, and $(x_{st} : s \geq 0)$ is a Brownian motion in M with drift q_{st}.

So far we have ignored what is happening to the starting point, but that is very simple. Previsibility forces $X_0(x)$ to be a function of the starting point x_0 alone, giving us a vector field on M, which we again denote X_0. Then x_{0t} obeys the autonomous equation
$$\partial_t x_{0t} = X_0(x_{0t}) \partial t.$$
If we assume that X_0 is C^1 say, then the law of x_{0t} is given by
$$0 = \frac{\partial}{\partial t} \mathbb{E}\big[f(x_{0t}) \exp\big\{ - \int_0^t \operatorname{div} X_0(x_{0\tau}) d\tau \big\} \big].$$

On the other hand, conditional on x_{0t}, the law of $(x_{st} : s \geq 0)$ is absolutely continuous with respect to $\mathbb{P}^{x_{0t}}$, at least on compact s-intervals, with density given by the Cameron–Martin formula. We have
$$\begin{aligned} &\frac{\partial}{\partial t} \Big\{ \int_0^\infty \langle q_{st}, d_s x_{st} \rangle - \tfrac{1}{2} \int_0^\infty |q_{st}|^2 ds \Big\} \\ &\quad = \int_0^\infty \langle \frac{D}{\partial t} q_{st}, d_s x_{st} \rangle \\ &\quad = \int_0^\infty \langle \left(\frac{D}{\partial s} - \tfrac{1}{2} \mathrm{Ricci} \right) X_{st}, d_s x_{st} \rangle. \end{aligned}$$

Hence the law of $x_t = (x_{st} : s \geq 0)$ is given by

$$0 = \frac{\partial}{\partial t}\mathbb{E}\big[F(x_t)\exp\big\{-\int_0^t \operatorname{div} X(x_\tau)d\tau\big\}\big]$$

for all bounded Borel functions F on Ω, where

$$\operatorname{div} X(x) = \operatorname{div} X_0(x_0) + \int_0^\infty \langle\left(\frac{D}{\partial s} - \tfrac{1}{2}\mathrm{Ricci}\right) X_s(x), dx_s\rangle.$$

This is our Cameron–Martin formula for the free path space. For suitably smooth functions F we can evaluate the derivative at $t = 0$ to recover the integration by parts formula of §2

$$\mathbb{E}\langle dF, X\rangle = \mathbb{E}[F\operatorname{div} X].$$

That concludes our rough argument.

The only serious gap in the above argument is the need to establish the existence of a flow for our vector field X, within the class of two-parameter semimartingales. Obviously, something in the nature of a Lipschitz condition looks desirable. But truly Lipschitz functions on Ω form an overly restricted class. We shall not attempt to find natural conditions on the vector field X, but restrict attention to the case already considered in §2. Let there be given C^2 vector fields $X_1, \dots, X_m$ on M, together with an absolutely continuous function $h_s = (h_s^1, \dots, h_s^m)$ in $\mathbb{R}^m$ satisfying

$$\int_0^\infty |\dot{h}_s|^2 ds < \infty. \tag{2}$$

Then for $\mathbb{P}$-almost all $x \in \Omega$ and all $s \geq 0$ we can define $X_s(x) \in T_{x_s}M$ by

$$X_s(x) = \tau_s \sum_{i=1}^m h_s^i X_i(x_0) \tag{3}$$

where τ_s denotes parallel translation $T_{x_0}M \to T_{x_s}M$ along x.

We state a special case of ([N2], Theorem 3.2.6) suited to our present needs.

Theorem 3.1. *Let M be a C^4 compact Riemannian manifold with Levi–Civita connection. Let*

$$\beta : OM \to TM \otimes T^*M$$

be a C^2 map of the fibres. Suppose we are given regular semimartingale boundary values $(x_{s0} : s \geq 0)$ and $(x_{0t} : t \geq 0)$ in M together with $u_{00} = v_{00} \in O_{x_{00}}M$. Then there exist unique two-parameter semimartingales x_{st} in M and u_{st}, v_{st} in OM over x_{st} such that $u_{s0} = v_{s0}$ and $u_{0t} = v_{0t}$, and satisfying

$$\begin{aligned} D_s\partial_t x_{st} &= \beta(u_{st})\partial_s x_{st}\partial t,\\ D_s u_{st} &= 0,\\ D_t v_{st} &= 0. \end{aligned}$$

We make some explanatory remarks. In this context regularity of the boundary values means uniformly Lipschitz quadratic variation and finite variation part. The auxilliary processes u_{st} and v_{st} are lifts of x_{st} in OM, which agree and are horizontal on the s and t-axes; then u_{st} is made horizontal along $(x_{st} : s \geq 0)$ for each $t \geq 0$, whereas v_{st} is horizontal along $(x_{st} : t \geq 0)$ for each $s \geq 0$. Parallel translation along $(x_{st} : s \geq 0)$ is then given by

$$\tau_{st} = u_{st} u_{0t}^{-1}.$$

The process v_{st} already appeared above in analysing the law of x_{st}.

In order to apply Theorem 3.1 to our present problem, we first integrate the C^2 vector field

$$X_0(x_0) = \sum_{i=1}^{m} h_0^i X_i(x_0)$$

which governs the autonomous motion of the base point

$$\dot{x}_{0t} = X_0(x_{0t}).$$

We denote by u_{0t} the horizontal lift along x_{0t} starting from u_{00}, and set

$$(k_t)_i = u_{0t}^{-1} X_i(x_{0t}),$$

k_t taking values in $(\mathbb{R}^m)^*$. The flow equation

$$\dot{x}_t = X(x_t) \tag{3}$$

is then equivalent to the system of two-parameter hyperbolic equations

$$\begin{aligned} D_s \partial_t x_{st} &= u_{st} k_t (\partial h_s) \partial t, \\ D_s u_{st} &= 0, \\ D_t v_{st} &= 0. \end{aligned}$$

In the case where h_s has bounded derivative and so is regular, we can now appeal to Theorem 3.1, applied to the augmented process $\tilde{x}_{st} = (x_{st}, h_{st}, k_{st})$ in $M \times \mathbb{R}^m \times (\mathbb{R}^m)^*$, with $h_{st} = h_s$ and $k_{st} = k_t$, satisfying

$$D_s \partial_t h_{st} = 0, \quad D_s \partial_t k_{st} = 0.$$

Hence (3) has a unique solution, whch is a two-parameter semimartingale. One can then pass to the case of general h_s by a time-change argument in s, as in ([N2], §4.2). Thus we obtain

Theorem 3.2. *Let $x_0 = (x_{s0} : s \geq 0)$ be a Brownian motion in M. Then there exists a unique two-parameter semimartingale $(x_{st} : s \geq 0, t \geq 0)$ satisfying*

$$\partial_t x_{st} = \tau_{st} \sum_{i=1}^{m} h_s^i X_i(x_0) \partial t.$$

The calculation of the law of x_{st} made above is now justified. The presence of the Ricci term in the drift means that (2) is not sufficient to make the law of $x_t = (x_{st} : s \geq 0)$ absolutely continuous with respect to $\mathbb{P}$, unless one restricts to compact s-intervals. The combination of (2) and

$$\int_0^\infty |h_s|^2 ds < \infty \tag{4}$$

is of course sufficient. We summarize our conclusions.

Theorem 3.3. *Let X be defined $\mathbb{P}$-almost everywhere on Ω by*

$$X_s(x) = \tau_s \sum_{i=1}^{m} h_s^i X_i(x_0)$$

where h_s satisfies (2) and (4) and $X_1, \dots, X_m$ are C^2 vector fields on M. There exists a unique two-parameter semimartingale $(x_{st} : s \geq 0, t \geq 0)$ such that the path-valued process $x_t = (x_{st} : s \geq 0)$ satisfies

(i) $x_0 = x$;
(ii) x_t *has law absolutely continuous with respect to* $\mathbb{P}$ *for all* $t \geq 0$;
(iii) $\dot{x}_t = X(x_t)$.

Moreover for every bounded Borel function F on Ω we have

$$0 = \frac{\partial}{\partial t} \mathbb{E}\big[F(x_t) \exp\big\{-\int_0^t \operatorname{div} X(x_\tau) d\tau\big\}\big]$$

where

$$\operatorname{div} X(x) = \operatorname{div} X_0(x_0) + \int_0^\infty \langle \left(\frac{D}{\partial s} - \tfrac{1}{2}\mathrm{Ricci} \right) X_s(x), dx_s \rangle.$$

Finally for every smooth cylinder function $F(x) = f(x_{s_1}, \dots, x_{s_k})$ we have the integration by parts formula

$$\mathbb{E}\langle dF, X \rangle = \mathbb{E}[F \operatorname{div} X]$$

where

$$\langle dF, X \rangle(x) = \sum_{j=1}^{k} \langle d_j f(x_{s_1}, \dots, x_{s_k}), X_{s_j}(x) \rangle.$$

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