1.12 Appendix: asymptotics for $n!$

Our analysis of recurrence and transience for random walks in Section 1.6 rested heavily on the use of the asymptotic relation

$$n! \sim A\sqrt{n}(n/e)^n \quad \text{as } n \to \infty$$

for some $A \in [1, \infty)$. Here is a derivation.

We make use of the power series expansions for $|t| < 1$

$$\begin{aligned} \log(1+t) &= t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \dots \\ \log(1-t) &= -t - \tfrac{1}{2}t^2 - \tfrac{1}{3}t^3 - \dots . \end{aligned}$$

By subtraction we obtain

$$\tfrac{1}{2}\log\left(\frac{1+t}{1-t}\right) = t + \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + \dots .$$

Set $A_n = n!/(n^{n+1/2}e^{-n})$ and $a_n = \log A_n$. Then, by a straightforward calculation

$$a_n - a_{n+1} = (2n+1)\frac{1}{2}\log\left(\frac{1+(2n+1)^{-1}}{1-(2n+1)^{-1}}\right) - 1.$$

By the series expansion written above we have

$$\begin{aligned} a_n - a_{n+1} &= (2n+1)\left\{\frac{1}{(2n+1)} + \frac{1}{3}\frac{1}{(2n+1)^3} + \frac{1}{5}\frac{1}{(2n+1)^5} + \dots\right\} - 1 \\ &= \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \dots \\ &\le \frac{1}{3}\left\{\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots\right\} \\ &= \frac{1}{3}\frac{1}{(2n+1)^2 - 1} = \frac{1}{12n} - \frac{1}{12(n+1)} . \end{aligned}$$

It follows that a_n decreases and $a_n - 1/(12n)$ increases as $n \to \infty$. Hence $a_n \to a$ for some $a \in [0, \infty)$ and hence $A_n \to A$, as $n \to \infty$, where $A = e^a$.