The average value of the walk is constant; indeed it has the stronger property that the average value of the walk at some future time is always simply the current value. In precise terms we have

$$\mathbb{E}X_n = \mathbb{E}X_0;$$

and the stronger property says that, for $n \geq m$,

$$\mathbb{E}(X_n - X_m \mid X_0 = i_0, \dots, X_m = i_m) = 0.$$

This stronger property says that $(X_n)_{n\geq 0}$ is in fact a martingale.

Here is the general definition. Let us fix for definiteness a Markov chain $(X_n)_{n\geq 0}$ and write $\mathcal{F}_n$ for the collection of all sets depending only on $X_0, \dots, X_n$. The sequence $(\mathcal{F}_n)_{n\geq 0}$ is called the *filtration* of $(X_n)_{n\geq 0}$ and we think of $\mathcal{F}_n$ as representing the state of knowledge, or history, of the chain up to time n. A process $(M_n)_{n\geq 0}$ is called *adapted* if M_n depends only on $X_0, \dots, X_n$. A process $(M_n)_{n\geq 0}$ is called *integrable* if $\mathbb{E}|M_n| < \infty$ for all n. An adapted integrable process $(M_n)_{n\geq 0}$ is called a *martingale* if

$$\mathbb{E}[(M_{n+1} - M_n)1_A] = 0$$

for all $A \in \mathcal{F}_n$ and all n. Since the collection $\mathcal{F}_n$ consists of countable unions of elementary events such as

$$\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\},$$

this martingale property is equivalent to saying that

$$\mathbb{E}(M_{n+1} - M_n \mid X_0 = i_0, \dots, X_n = i_n) = 0$$

for all $i_0, \dots, i_n$ and all n.

A third formulation of the martingale property involves another notion of conditional expectation. Given an integrable random variable Y, we define

$$\mathbb{E}(Y \mid \mathcal{F}_n) = \sum_{i_0, \dots, i_n} \mathbb{E}(Y \mid X_0 = i_0, \dots, X_n = i_n)1_{\{X_0 = i_0, \dots, X_n = i_n\}}.$$

The random variable $\mathbb{E}(Y \mid \mathcal{F}_n)$ is called the *conditional expectation* of Y given $\mathcal{F}_n$. In passing from Y to $\mathbb{E}(Y \mid \mathcal{F}_n)$, what we do is to replace on each elementary event $A \in \mathcal{F}_n$, the random variable Y by its average value $\mathbb{E}(Y \mid A)$. It is easy to check that an adapted integrable process $(M_n)_{n\geq 0}$ is a martingale if and only if

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n \quad \text{for all } n.$$

Conditional expectation is a partial averaging, so if we complete the process and average the conditional expectation we should get the full expectation

$$\mathbb{E}\big(\mathbb{E}(Y \mid \mathcal{F}_n)\big) = \mathbb{E}(Y).$$

It is easy to check that this formula holds.

In particular, for a martingale

$$\mathbb{E}(M_n) = \mathbb{E}\big(\mathbb{E}(M_{n+1} \mid \mathcal{F}_n)\big) = \mathbb{E}(M_{n+1})$$

so, by induction

$$\mathbb{E}(M_n) = \mathbb{E}(M_0).$$

This was already clear on taking $A = \Omega$ in our original definition of a martingale.

We shall prove one general result about martingales, then see how it explains some things we know about the simple symmetric random walk. Recall that a random variable

$$T : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$$

is a *stopping time* if $\{T = n\} \in \mathcal{F}_n$ for all $n < \infty$. An equivalent condition is that $\{T \le n\} \in \mathcal{F}_n$ for all $n < \infty$. Recall from Section 1.4 that all sorts of hitting times are stopping times.

Theorem 4.1.1 (Optional stopping theorem). *Let $(M_n)_{n\ge 0}$ be a martingale and let T be a stopping time. Suppose that at least one of the following conditions holds:*

(i) *$T \le n$ for some n;*

(ii) *$T < \infty$ and $|M_n| \le C$ whenever $n \le T$.*

Then $\mathbb{E}M_T = \mathbb{E}M_0$.

Proof. Assume that (i) holds. Then

$$\begin{aligned} M_T - M_0 &= (M_T - M_{T-1}) + \ldots + (M_1 - M_0) \\ &= \sum_{k=0}^{n-1} (M_{k+1} - M_k) 1_{k<T}. \end{aligned}$$

Now $\{k < T\} = \{T \le k\}^c \in \mathcal{F}_k$ since T is a stopping time, and so

$$\mathbb{E}[(M_{k+1} - M_k) 1_{k<T}] = 0$$

since $(M_k)_{k\ge 0}$ is a martingale. Hence

$$\mathbb{E}M_T - \mathbb{E}M_0 = \sum_{k=0}^{n-1} \mathbb{E}[(M_{k+1} - M_k) 1_{k<T}] = 0.$$

If we do not assume (i) but (ii), then the preceding argument applies to the stopping time $T \wedge n$, so that $\mathbb{E}M_{T\wedge n} = \mathbb{E}M_0$. Then

$$|\mathbb{E}M_T - \mathbb{E}M_0| = |\mathbb{E}M_T - \mathbb{E}M_{T\wedge n}| \le \mathbb{E}|M_T - M_{T\wedge n}| \le 2C\mathbb{P}(T > n)$$

for all n. But $\mathbb{P}(T > n) \to 0$ as $n \to \infty$, so $\mathbb{E}M_T = \mathbb{E}M_0$. $\square$

Returning to the simple symmetric random walk $(X_n)_{n\geq 0}$, suppose that $X_0 = 0$ and we take

$$T = \inf\{n \geq 0 : X_n = -a \quad \text{or} \quad X_n = b\}$$

where $a, b \in \mathbb{N}$ are given. Then T is a stopping time and $T < \infty$ by recurrence of finite closed classes. Thus condition (ii) of the optional stopping theorem applies with $M_n = X_n$ and $C = a \vee b$. We deduce that $\mathbb{E}X_T = \mathbb{E}X_0 = 0$. So what? Well, now we can compute

$$p = \mathbb{P}(X_n \text{ hits } -a \text{ before } b).$$

We have $X_T = -a$ with probability p and $X_T = b$ with probability $1 - p$, so

$$0 = \mathbb{E}X_T = p(-a) + (1-p)b$$

giving

$$p = b/(a+b).$$

There is an entirely different, Markovian, way to compute p, using the methods of Section 1.4. But the intuition behind the result $\mathbb{E}X_T = 0$ is very clear: a gambler, playing a fair game, leaves the casino once losses reach a or winnings reach b, whichever is sooner; since the game is fair, the average gain should be zero.

We discussed in Section 1.3 the counter-intuitive case of a gambler who keeps on playing a fair game against an infinitely rich casino, with the certain outcome of ruin. This game ends at the finite stopping time

$$T = \inf\{n \geq 0 : X_n = -a\}$$

where a is the gambler's initial fortune. Since $X_T = -a$ we have

$$\mathbb{E}X_T = -a \neq 0 = \mathbb{E}X_0$$

but this does not contradict the optional stopping theorem because neither condition (i) nor condition (ii) is satisfied. Thus, while intuition might suggest that $\mathbb{E}X_T = \mathbb{E}X_0$ is rather obvious, some care is needed as it is not always true.

The example just discussed was rather special in that the chain $(X_n)_{n\geq 0}$ itself was a martingale. Obviously, this is not true in general; indeed a martingale is necessarily real-valued and we do not in general insist that the state-space I is contained in $\mathbb{R}$. Nevertheless, to every Markov chain is associated a whole collection of martingales, and these martingales characterize the chain. This is the basis of a deep connection between martingales and Markov chains.

We recall that, given a function $f : I \to \mathbb{R}$ and a Markov chain $(X_n)_{n\geq 0}$ with transition matrix P, we have

$$(P^n f)(i) = \sum_{j\in I} p_{ij}^{(n)} f_j = \mathbb{E}_i\big(f(X_n)\big).$$

Theorem 4.1.2. *Let $(X_n)_{n\geq 0}$ be a random process with values in I and let P be a stochastic matrix. Write $(\mathcal{F}_n)_{n\geq 0}$ for the filtration of $(X_n)_{n\geq 0}$. Then the following are equivalent:*

(i) *$(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P;*

(ii) *for all bounded functions $f : I \to \mathbb{R}$, the following process is a martingale:*

$$M_n^f = f(X_n) - f(X_0) - \sum_{m=0}^{n-1} (P - I) f(X_m).$$

Proof. Suppose (i) holds. Let f be a bounded function. Then

$$|(Pf)(i)| = |\sum_{j\in I} p_{ij} f_j| \leq \sup_j |f_j|$$

so

$$|M_n^f| \leq 2(n+1) \sup_j |f_j| < \infty$$

showing that M_n^f is integrable for all n.

Let $A = \{X_0 = i_0, \dots, X_n = i_n\}$. By the Markov property

$$\mathbb{E}\big(f(X_{n+1}) \mid A\big) = \mathbb{E}_{i_n}\big(f(X_1)\big) = (Pf)(i_n)$$

so

$$\mathbb{E}(M_{n+1}^f - M_n^f \mid A) = \mathbb{E}[f(X_{n+1}) - (Pf)(X_n) \mid A] = 0$$

and so $(M_n^f)_{n\geq 0}$ is a martingale.

On the other hand, if (ii) holds, then

$$\mathbb{E}[f(X_{n+1}) - (Pf)(X_n) \mid X_0 = i_0, \dots, X_n = i_n] = 0$$

for all bounded functions f. On taking $f = 1_{\{i_{n+1}\}}$ we obtain

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) = p_{i_n i_{n+1}}$$

so $(X_n)_{n\geq 0}$ is Markov with transition matrix P. $\square$

Some more martingales associated to a Markov chain are described in the next result. Notice that we drop the requirement that f be bounded.

Theorem 4.1.3. *Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. Suppose that a function $f : \mathbb{N} \times I \to \mathbb{R}$ satisfies, for all $n \geq 0$, both*

$$\mathbb{E}|f(n, X_n)| < \infty$$

and

$$(Pf)(n+1, i) = \sum_{j\in I} p_{ij} f(n+1, j) = f(n, i).$$

Then $M_n = f(n, X_n)$ is a martingale.

Proof. We have assumed that M_n is integrable for all n. Then, by the Markov property

$$\begin{aligned}\mathbb{E}(M_{n+1} - M_n \mid X_0 = i_0, \dots, X_n = i_n) &= \mathbb{E}_{i_n}[f(n+1, X_1) - f(n, X_0)] \\ &= (Pf)(n+1, i_n) - f(n, i_n) = 0.\end{aligned}$$

So $(M_n)_{n\geq 0}$ is a martingale. □

Let us see how this theorem works in the case where $(X_n)_{n\geq 0}$ is a simple random walk on $\mathbb{Z}$, starting from 0. We consider $f(i) = i$ and $g(n,i) = i^2 - n$. Since $|X_n| \leq n$ for all n, we have

$$\mathbb{E}|f(X_n)| < \infty, \quad \mathbb{E}|g(n, X_n)| < \infty.$$

Also

$$\begin{aligned}&(Pf)(i) = (i-1)/2 + (i+1)/2 = i = f(i), \\ &(Pg)(n+1, i) = (i-1)^2/2 + (i-1)^2/2 - (n+1) = i^2 - n = g(n,i).\end{aligned}$$

Hence both $X_n = f(X_n)$ and $Y_n = g(n, X_n)$ are martingales.

In order to put this to some use, consider again the stopping time

$$T = \inf\{n \geq 0 : X_n = -a \quad \text{or} \quad X_n = b\}$$

where $a, b \in \mathbb{N}$. By the optional stopping theorem

$$0 = \mathbb{E}(Y_0) = \mathbb{E}(Y_{T\wedge n}) = \mathbb{E}(X_{T\wedge n}^2) - \mathbb{E}(T \wedge n).$$

Hence

$$\mathbb{E}(T \wedge n) = \mathbb{E}(X_{T\wedge n}^2).$$

On letting $n \to \infty$, the left side converges to $\mathbb{E}(T)$, by monotone convergence, and the right side to $\mathbb{E}(X_T^2)$ by bounded convergence. So we obtain

$$\mathbb{E}(T) = \mathbb{E}(X_T^2) = a^2 p + b^2(1-p) = ab.$$

We have given only the simplest examples of the use of martingales in studying Markov chains. Some more will appear in later sections. For an excellent introduction to martingales and their applications we recommend *Probability with Martingales* by David Williams (Cambridge University Press, 1991).

Exercise

4.1.1 Let $(X_n)_{n\geq 0}$ be a Markov chain on I and let A be an absorbing set in I. Set

$$T = \inf\{n \geq 0 : X_n \in A\}$$

and

$$h_i = \mathbb{P}_i(X_n \in A \text{ for some } n \geq 0) = \mathbb{P}_i(T < \infty).$$

Show that $M_n = h(X_n)$ is a martingale.

4.2 Potential theory

Several physical theories share a common mathematical framework, which is known as potential theory. One example is Newton's theory of gravity, but potential theory is also relevant to electrostatics, fluid flow and the diffusion of heat. In gravity, a distribution of mass, of density ρ say, gives rise to a gravitational potential ϕ, which in suitable units satisfies the equation

$$-\Delta\phi = \rho,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. The potential ϕ is felt physically through its gradient

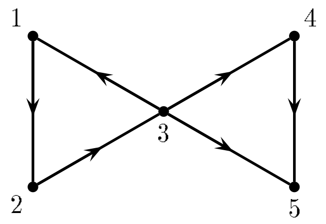
$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

which gives the force of gravity acting on a particle of unit mass. Markov chains, where space is discrete, obviously have no direct link with this theory, in which space is a continuum. An indirect link is provided by Brownian motion, which we shall discuss in Section 4.4.

In this section we are going to consider *potential theory for a countable state-space*, which has much of the structure of the continuum version. This discrete theory amounts to doing Markov chains without the probability, which has the disadvantage that one loses the intuitive picture of the process, but the advantage of wider applicability. We shall begin by introducing the idea of potentials associated to a Markov chain, and by showing how to calculate these potentials. This is a unifying idea, containing within it other notions previously considered such as hitting probabilities and expected hitting times. It also finds application when one associates costs to Markov chains in modelling economic activity: see Section 5.4.

Once we have established the basic link between a Markov chain and its associated potentials, we shall briefly run through some of the main features of potential theory, explaining their significance in terms of Markov chains. This is the easiest way to appreciate the general structure of potential theory, unobscured by technical difficulties. The basic ideas of boundary theory for Markov chains will also be introduced.

Before we embark on a general discussion of potentials associated to a Markov chain, here are two simple examples. In these examples the potential ϕ has the interpretation of expected total cost.



Example 4.2.1

Consider the discrete-time random walk on the directed graph shown above, which at each step choses among the allowable transitions with equal probability. Suppose that on each visit to states $i = 1, 2, 3, 4$ a cost c_i is incurred, where $c_i = i$. What is the fair price to move from state 3 to state 4?

The fair price is always the difference in the expected total cost. We denote by ϕ_i the expected total cost starting from i. Obviously, $\phi_5 = 0$ and by considering the effect of a single step we see that

$$\begin{aligned}
\phi_1 &= 1 + \phi_2, \\
\phi_2 &= 2 + \phi_3, \\
\phi_3 &= 3 + \tfrac{1}{3}\phi_1 + \tfrac{1}{3}\phi_4, \\
\phi_4 &= 4.
\end{aligned}$$

Hence $\phi_3 = 8$ and the fair price to move from 3 to 4 is 4.

We shall now consider two variations on this problem. First suppose our process is, instead, the continuous-time random walk $(X_t)_{t\geq 0}$ on the same directed graph which makes each allowable transition at rate 1, and suppose cost is incurred at rate $c_i = i$ in state i for $i = 1, 2, 3, 4$. Thus the total cost is now

$$\int_0^\infty c(X_s)ds.$$

What now is the fair price to move from 3 to 4? The expected cost incurred on each visit to i is given by c_i/q_i and $q_1 = 1, q_2 = 1, q_3 = 3, q_4 = 1$. So we see, as before

$$\begin{aligned}
\phi_1 &= 1 + \phi_2, \\
\phi_2 &= 2 + \phi_3, \\
\phi_3 &= \tfrac{3}{3} + \tfrac{1}{3}\phi_1 + \tfrac{1}{3}\phi_4, \\
\phi_4 &= 4.
\end{aligned}$$

Hence $\phi_3 = 5$ and the fair price to move from 3 to 4 is 1.

In the second variation we consider the discrete-time random walk $(X_n)_{n\geq 0}$ on the modified graph shown below. Where there is no arrow, transitions are allowed