(b) (infinitesimal definition) *for all* $t, h \geq 0$, *conditional on* $X_t = i$, X_{t+h} *is independent of* $(X_s : s \leq t)$ *and, as* $h \downarrow 0$, *uniformly in* t, *for all* j

$$\mathbb{P}(X_{t+h} = j \mid X_t = i) = \delta_{ij} + q_{ij}h + o(h);$$

(c) (transition probability definition) *for all* $n = 0, 1, 2, \dots$, *all times* $0 \leq t_0 \leq t_1 \leq \dots \leq t_{n+1}$ *and all states* $i_0, \dots, i_{n+1}$

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n) = p_{i_n i_{n+1}}(t_{n+1} - t_n)$$

where $(p_{ij}(t) : i, j \in I, t \geq 0)$ *is the solution of the forward equation*

$$P'(t) = P(t)Q, \quad P(0) = I.$$

If $(X_t)_{t \geq 0}$ satisfies any of these conditions then it is called a *Markov chain with generator matrix* Q. We say that $(X_t)_{t \geq 0}$ is *Markov*(λ, Q) for short, where λ is the distribution of X_0.

Proof. (a) $\Rightarrow$ (b) Suppose (a) holds, then, as $h \downarrow 0$,

$$\mathbb{P}_i(X_h = i) \geq \mathbb{P}_i(J_1 > h) = e^{-q_i h} = 1 + q_{ii}h + o(h)$$

and for $j \neq i$ we have

$$\begin{aligned}\mathbb{P}_i(X_h = j) &\geq \mathbb{P}_(J_1 \leq h, Y_1 = j, S_2 > h) \\ &= (1 - e^{-q_i h})\pi_{ij} e^{-q_j h} = q_{ij}h + o(h).\end{aligned}$$

Thus for every state j there is an inequality

$$\mathbb{P}_i(X_h = j) \geq \delta_{ij} + q_{ij}h + o(h)$$

and by taking the finite sum over j we see that these must in fact be equalities. Then by the Markov property, for any $t, h \geq 0$, conditional on $X_t = i$, X_{t+h} is independent of $(X_s : s \leq t)$ and, as $h \downarrow 0$, uniformly in t

$$\mathbb{P}(X_{t+h} = j \mid X_t = i) = \mathbb{P}_i(X_h = j) = \delta_{ij} + q_{ij}h + o(h).$$

(b) $\Rightarrow$ (c) Set $p_{ij}(t) = \mathbb{P}_i(X_t = j) = \mathbb{P}(X_t = j \mid X_0 = i)$. If (b) holds, then for all $t, h \geq 0$, as $h \downarrow 0$, uniformly in t

$$\begin{aligned}p_{ij}(t+h) &= \sum_{k \in I} \mathbb{P}_i(X_t = k)\mathbb{P}(X_{t+h} = j \mid X_t = k) \\ &= \sum_{k \in I} p_{ik}(t)(\delta_{kj} + q_{kj}h + o(h)).\end{aligned}$$

Since I is finite we have

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \in I} p_{ik}(t)q_{kj} + O(h)$$

so, letting $h \downarrow 0$, we see that $p_{ij}(t)$ is differentiable on the right. Then by uniformity we can replace t by $t-h$ in the above and let $h \downarrow 0$ to see first that $p_{ij}(t)$ is continuous on the left, then differentiable on the left, hence differentiable, and satisfies the forward equations

$$p'_{ij}(t) = \sum_{k \in I} p_{ik}(t) q_{kj}, \quad p_{ij}(0) = \delta_{ij}.$$

Since I is finite, $p_{ij}(t)$ is then the unique solution by Theorem 2.1.1. Also, if (b) holds, then

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n) = \mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_n} = i_n)$$

and, moreover, (b) holds for $(X_{t_n+t})_{t \geq 0}$ so, by the above argument,

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_n} = i_n) = p_{i_n i_{n+1}}(t_{n+1} - t_n),$$

proving (c).

(c) $\Rightarrow$ (a) See the proof of Theorem 2.4.3. $\square$

We know from Theorem 2.1.1 that for I finite the forward and backward equations have the same solution. So in condition (c) of the result just proved we could replace the forward equation with the backward equation. Indeed, there is a slight variation of the argument from (b) to (c) which leads directly to the backward equation.

The deduction of (c) from (b) above can be seen as the matrix version of the following result: for $q \in \mathbb{R}$ we have

$$\left(1 + \frac{q}{n} + o\left(\frac{1}{n}\right)\right)^n \to e^q \qquad \text{as } n \to \infty.$$

Suppose (b) holds and set

$$p_{ij}(t, t+h) = \mathbb{P}(X_{t+h} = j \mid X_t = i);$$

then $P(t, t+h) = (p_{ij}(t, t+h) : i, j \in I)$ satisfies

$$P(t, t+h) = I + Qh + o(h)$$

and

$$P(0,t) = P\left(0, \frac{t}{n}\right) P\left(\frac{t}{n}, \frac{2t}{n}\right) \dots P\left(\frac{(n-1)t}{n}, t\right) = \left(I + \frac{tQ}{n} + o\left(\frac{1}{n}\right)\right)^n.$$

Some care is needed in making this precise, since the $o(h)$ terms, though uniform in t, are not *a priori* identical. On the other hand, in (c) we see that

$$P(0,t) = e^{tQ}.$$

We turn now to the case of infinite state-space. The backward equation may still be written in the form

$$P'(t) = QP(t), \quad P(0) = I$$

only now we have an infinite system of differential equations

$$p'_{ij}(t) = \sum_{k \in I} q_{ik} p_{kj}(t), \quad p_{ij}(0) = \delta_{ij}$$

and the results on matrix exponentials given in Section 2.1 no longer apply. A solution to the backward equation is any matrix $(p_{ij}(t) : i, j \in I)$ of differentiable functions satisfying this system of differential equations.

Theorem 2.8.3. *Let Q be a Q-matrix. Then the backward equation*

$$P'(t) = QP(t), \quad P(0) = I$$

has a minimal non-negative solution $(P(t) : t \geq 0)$. This solution forms a matrix semigroup

$$P(s)P(t) = P(s+t) \qquad \text{for all } s, t \geq 0.$$

We shall prove this result by a probabilistic method in combination with Theorem 2.8.4. Note that if I is finite we must have $P(t) = e^{tQ}$ by Theorem 2.1.1. We call $(P(t) : t \geq 0)$ the *minimal non-negative semigroup* associated to Q, or simply the *semigroup* of Q, the qualifications *minimal* and *non-negative* being understood.

Here is the key result for Markov chains with infinite state-space. There are just two alternative definitions now as the infinitesimal characterization become problematic for infinite state-space.

Theorem 2.8.4. *Let $(X_t)_{t\geq 0}$ be a minimal right-continuous process with values in I. Let Q be a Q-matrix on I with jump matrix Π and semigroup $(P(t) : t \geq 0)$. Then the following conditions are equivalent:*

(a) (jump chain/holding time definition) *conditional on $X_0 = i$, the jump chain $(Y_n)_{n\geq 0}$ of $(X_t)_{t\geq 0}$ is discrete-time Markov(δ_i, Π) and for each $n \geq 1$, conditional on $Y_0, \dots, Y_{n-1}$, the holding times $S_1, \dots, S_n$ are independent exponential random variables of parameters $q(Y_0), \dots, q(Y_{n-1})$ respectively;*

(b) (transition probability definition) *for all $n = 0, 1, 2, \dots$, all times $0 \leq t_0 \leq t_1 \leq \dots \leq t_{n+1}$ and all states $i_0, i_1, \dots, i_{n+1}$*

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n) = p_{i_n i_{n+1}}(t_{n+1} - t_n).$$

If $(X_t)_{t\geq 0}$ satisfies any of these conditions then it is called a *Markov chain with generator matrix* Q. We say that $(X_t)_{t\geq 0}$ is *Markov*(λ, Q) for short, where λ is the distribution of X_0.

Proof of Theorems 2.8.3 and 2.8.4. We know that there exists a process $(X_t)_{t\geq 0}$ satisfying (a). So let us *define* $P(t)$ by

$$p_{ij}(t) = \mathbb{P}_i(X_t = j).$$

Step 1. *We show that $P(t)$ satisfies the backward equation.*

Conditional on $X_0 = i$ we have $J_1 \sim E(q_1)$ and $X_{J_1} \sim (\pi_{ik} : k \in I)$. Then conditional on $J_1 = s$ and $X_{J_1} = k$ we have $(X_{s+t})_{t\geq 0} \sim$ Markov(δ_k, Q). So

$$\mathbb{P}_i(X_t = j, t < J_1) = e^{-q_i t}\delta_{ij}$$

and

$$\mathbb{P}_i(J_1 \leq t, X_{J_1} = k, X_t = j) = \int_0^t q_i e^{-q_i s} \pi_{ik} p_{kj}(t-s)ds.$$

Therefore

$$\begin{aligned} p_{ij}(t) &= \mathbb{P}_i(X_t = j, t < J_1) + \sum_{k \neq i} \mathbb{P}_i(J_1 \leq t, X_{J_1} = k, X_t = j) \\ &= e^{-q_i t}\delta_{ij} + \sum_{k \neq i} \int_0^t q_i e^{-q_i s} \pi_{ik} p_{kj}(t-s)ds. \end{aligned} \tag{2.1}$$

Make a change of variable $u = t - s$ in each of the integrals, interchange sum and integral by monotone convergence and multiply by $e^{q_i t}$ to obtain

$$e^{q_i t} p_{ij}(t) = \delta_{ij} + \int_0^t \sum_{k \neq i} q_i e^{q_i u} \pi_{ik} p_{kj}(u) du. \tag{2.2}$$

This equation shows, firstly, that $p_{ij}(t)$ is continuous in t for all i, j. Secondly, the integrand is then a uniformly converging sum of continuous functions, hence continuous, and hence $p_{ij}(t)$ is differentiable in t and satisfies

$$e^{q_i t}(q_i p_{ij}(t) + p'_{ij}(t)) = \sum_{k \neq i} q_i e^{q_i t} \pi_{ik} p_{kj}(t).$$

Recall that $q_i = -q_{ii}$ and $q_{ik} = q_i \pi_{ik}$ for $k \neq i$. Then, on rearranging, we obtain

$$p'_{ij}(t) = \sum_{k \in I} q_{ik} p_{kj}(t) \tag{2.3}$$

so $P(t)$ satisfies the backward equation.

The integral equation (2.1) is called the *integral form of the backward equation.*

Step 2. *We show that if $\widetilde{P}(t)$ is another non-negative solution of the backward equation, then $P(t) \leq \widetilde{P}(t)$, hence $P(t)$ is the minimal non-negative solution.*

The argument used to prove (2.1) also shows that

$$\begin{aligned} &\mathbb{P}_i(X_t = j, t < J_{n+1}) \\ &\quad = e^{-q_i t}\delta_{ij} + \sum_{k \neq i} \int_0^t q_i e^{-q_i s} \pi_{ik} \mathbb{P}_k(X_{t-s} = j, t - s < J_n) ds. \end{aligned} \tag{2.4}$$

On the other hand, if $\widetilde{P}(t)$ satisfies the backward equation, then, by reversing the steps from (2.1) to (2.3), it also satisfies the integral form:

$$\widetilde{p}_{ij}(t) = e^{-q_i t}\delta_{ij} + \sum_{k \neq i} \int_0^t q_i e^{-q_i s} \pi_{ik} \widetilde{p}_{kj}(t-s) ds. \tag{2.5}$$

If $\widetilde{P}(t) \geq 0$, then

$$\mathbb{P}_i(X_t = j, t < J_0) = 0 \leq \widetilde{p}_{ij}(t) \qquad \text{for all } i, j \text{ and } t.$$

Let us suppose inductively that

$$\mathbb{P}_i(X_t = j, t < J_n) \le \widetilde{p}_{ij}(t) \qquad \text{for all } i, j \text{ and } t,$$

then by comparing (2.4) and (2.5) we have

$$\mathbb{P}_i(X_t = j, t < J_{n+1}) \le \widetilde{p}_{ij}(t) \qquad \text{for all } i, j \text{ and } t,$$

and the induction proceeds. Hence

$$p_{ij}(t) = \lim_{n\to\infty} \mathbb{P}_i(X_t = j, t < J_n) \le \widetilde{p}_{ij}(t) \qquad \text{for all } i, j \text{ and } t.$$

Step 3. Since $(X_t)_{t\ge 0}$ does not return from ∞ we have

$$\begin{aligned} p_{ij}(s+t) = \mathbb{P}_i(X_{s+t} = j) &= \sum_{k\in I} \mathbb{P}_i(X_{s+t} = j \mid X_s = k)\mathbb{P}_i(X_s = k) \\ &= \sum_{k\in I} \mathbb{P}_i(X_s = k)\mathbb{P}_k(X_t = j) = \sum_{k\in I} p_{ik}(s)p_{kj}(t) \end{aligned}$$

by the Markov property. Hence $(P(t) : t \ge 0)$ is a matrix semigroup. This completes the proof of Theorem 2.8.3.

Step 4. Suppose, as we have throughout, that $(X_t)_{t\ge 0}$ satisfies (a). Then, by the Markov property

$$\begin{aligned} &\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n) \\ &\qquad\qquad = \mathbb{P}_{i_n}(X_{t_{n+1}-t_n} = i_{n+1}) = p_{i_n i_{n+1}}(t_{n+1} - t_n) \end{aligned}$$

so $(X_t)_{t\ge 0}$ satisfies (b). We complete the proof of Theorem 2.8.4 by the usual argument that (b) must now imply (a) (see the proof of Theorem 2.4.3, (c) $\Rightarrow$ (a)). $\square$

So far we have said nothing about the forward equation in the case of infinite state-space. Remember that the finite state-space results of Section 2.1 are no longer valid. The forward equation may still be written

$$P'(t) = P(t)Q, \quad P(0) = I,$$

now understood as an infinite system of differential equations

$$p'_{ij}(t) = \sum_{k\in I} p_{ik}(t)q_{kj}, \quad p_{ij}(0) = \delta_{ij}.$$

A solution is then any matrix $\big(p_{ij}(t) : i, j \in I\big)$ of differentiable functions satisfying this system of equations. We shall show that the semigroup $\big(P(t) : t \ge 0\big)$ of Q does satisfy the forward equations, by a probabilistic argument resembling Step 1 of the proof of Theorems 2.8.3 and 2.8.4. This time, instead of conditioning on the first event, we condition on the last event before time t. The argument is a little longer because there is no reverse-time Markov property to give the conditional distribution. We need the following time-reversal identity, a simple version of which was given in Theorem 2.3.4.

Lemma 2.8.5. *We have*

$$\begin{aligned} &q_{i_n}\mathbb{P}(J_n \le t < J_{n+1} \mid Y_0 = i_0, Y_1 = i_1, \ldots, Y_n = i_n) \\ &\quad = q_{i_0}\mathbb{P}(J_n \le t < J_{n+1} \mid Y_0 = i_n, \ldots, Y_{n-1} = i_1, Y_n = i_0). \end{aligned}$$

Proof. Conditional on $Y_0 = i_0, \ldots, Y_n = i_n$, the holding times $S_1, \ldots, S_{n+1}$ are independent with $S_k \sim E(q_{i_{k-1}})$. So the left-hand side is given by

$$\int_{\Delta(t)} q_{i_n} \exp\{-q_{i_n}(t - s_1 - \ldots - s_n)\} \prod_{k=1}^{n} q_{i_{k-1}} \exp\{-q_{i_{k-1}} s_k\} ds_k$$

where $\Delta(t) = \{(s_1, \ldots, s_n) : s_1 + \ldots + s_n \le t \text{ and } s_1, \ldots, s_n \ge 0\}$. On making the substitutions $u_1 = t - s_1 - \ldots - s_n$ and $u_k = s_{n-k+2}$, for $k = 2, \ldots, n$, we obtain

$$\begin{aligned} &q_{i_n}\mathbb{P}(J_n \le t < J_{n+1} \mid Y_0 = i_0, \ldots, Y_n = i_n) \\ &= \int_{\Delta(t)} q_{i_0} \exp\{-q_{i_0}(t - u_1 - \ldots - u_n)\} \prod_{k=1}^{n} q_{i_{n-k+1}} \exp\{-q_{i_{n-k+1}} u_k\} du_k \\ &= q_{i_0}\mathbb{P}(J_n \le t < J_{n+1} \mid Y_0 = i_n, \ldots, Y_{n-1} = i_1, Y_n = i_0). \end{aligned}$$ □

Theorem 2.8.6. *The minimal non-negative solution $\big(P(t) : t \ge 0\big)$ of the backward equation is also the minimal non-negative solution of the forward equation*

$$P'(t) = P(t)Q, \quad P(0) = I.$$

Proof. Let $(X_t)_{t\ge 0}$ denote the minimal Markov chain with generator matrix Q. By Theorem 2.8.4

$$\begin{aligned} p_{ij}(t) &= \mathbb{P}_i(X_t = j) \\ &= \sum_{n=0}^{\infty} \sum_{k \ne j} \mathbb{P}_i(J_n \le t < J_{n+1}, Y_{n-1} = k, Y_n = j). \end{aligned}$$

Now by Lemma 2.8.5, for $n \ge 1$, we have

$$\begin{aligned} &\mathbb{P}_i(J_n \le t < J_{n+1} \mid Y_{n-1} = k, Y_n = j) \\ &\quad = (q_i/q_j)\mathbb{P}_j(J_n \le t < J_{n+1} \mid Y_1 = k, Y_n = i) \\ &\quad = (q_i/q_j) \int_0^t q_j e^{-q_j s} \mathbb{P}_k(J_{n-1} \le t - s < J_n \mid Y_{n-1} = i) ds \\ &\quad = q_i \int_0^t e^{-q_j s} (q_k/q_i) \mathbb{P}_i(J_{n-1} \le t - s < J_n \mid Y_{n-1} = k) ds \end{aligned}$$

where we have used the Markov property of $(Y_n)_{n\geq 0}$ for the second equality. Hence

$$\begin{aligned}
p_{ij}(t) &= \delta_{ij}e^{-q_i t} + \sum_{n=1}^{\infty}\sum_{k\neq j}\int_0^t \mathbb{P}_i(J_{n-1} \leq t-s < J_n \mid Y_{n-1}=k) \\
&\qquad\qquad\qquad\qquad \times \mathbb{P}_i(Y_{n-1}=k, Y_n=j)q_k e^{-q_j s}ds \\
&= \delta_{ij}e^{-q_i t} + \sum_{n=1}^{\infty}\sum_{k\neq j}\int_0^t \mathbb{P}_i(J_{n-1} \leq t-s < J_n, Y_{n-1}=k)q_k \pi_{kj} e^{-q_j s}ds \\
&= \delta_{ij}e^{-q_i t} + \int_0^t \sum_{k\neq j} p_{ik}(t-s)q_{kj}e^{-q_j s}ds \qquad (2.6)
\end{aligned}$$

where we have used monotone convergence to interchange the sum and integral at the last step. This is the *integral form of the forward equation*. Now make a change of variable $u = t - s$ in the integral and multiply by $e^{q_j t}$ to obtain

$$p_{ij}(t)e^{q_j t} = \delta_{ij} + \int_0^t \sum_{k\neq j} p_{ik}(u)q_{kj}e^{q_j u}du. \qquad (2.7)$$

We know by equation (2.2) that $e^{q_i t}p_{ik}(t)$ is *increasing* for all i, k. Hence either

$$\sum_{k\neq j} p_{ik}(u)q_{kj} \quad \text{converges uniformly for} \quad u \in [0,t]$$

or

$$\sum_{k\neq j} p_{ik}(u)q_{kj} = \infty \quad \text{for all } u \geq t.$$

The latter would contradict (2.7) since the left-hand side is finite for all t, so it is the former which holds. We know from the backward equation that $p_{ij}(t)$ is continuous for all i, j; hence by uniform convergence the integrand in (2.7) is continuous and we may differentiate to obtain

$$p'_{ij}(t) + p_{ij}(t)q_j = \sum_{k\neq j} p_{ik}(t)q_{kj}.$$

Hence $P(t)$ solves the forward equation.

To establish minimality let us suppose that $\widetilde{p}_{ij}(t)$ is another solution of the forward equation; then we also have

$$\widetilde{p}_{ij}(t) = \delta_{ij}e^{-q_i t} + \sum_{k\neq j}\int_0^t \widetilde{p}_{ik}(t-s)q_{kj}e^{-q_j s}ds.$$

A small variation of the argument leading to (2.6) shows that, for $n \geq 0$

$$\begin{aligned}
&\mathbb{P}_i(X_t = j, t < J_{n+1}) \\
&\quad = \delta_{ij}e^{-q_i t} + \sum_{k\neq j}\int_0^t \mathbb{P}_i(X_t = j, t < J_n)q_{kj}e^{-q_j s}ds. \qquad (2.8)
\end{aligned}$$

If $\widetilde{P}(t) \geq 0$, then

$$\mathbb{P}(X_t = j, t < J_0) = 0 \leq \widetilde{p}_{ij}(t) \quad \text{for all } i, j \text{ and } t.$$

Let us suppose inductively that

$$\mathbb{P}_i(X_t = j, t < J_n) \leq \widetilde{p}_{ij}(t) \quad \text{for all } i, j \text{ and } t;$$

then by comparing (2.7) and (2.8) we obtain

$$\mathbb{P}_i(X_t = j, t < J_{n+1}) \leq \widetilde{p}_{ij}(t) \quad \text{for all } i, j \text{ and } t$$

and the induction proceeds. Hence

$$p_{ij}(t) = \lim_{n \to \infty} \mathbb{P}_i(X_t = j, t < J_n) \leq \widetilde{p}_{ij}(t) \quad \text{for all } i, j \text{ and } t. \qquad \square$$

Exercises

2.8.1 Two fleas are bound together to take part in a nine-legged race on the vertices A, B, C of a triangle. Flea 1 hops at random times in the clockwise direction; each hop takes the pair from one vertex to the next and the times between successive hops of Flea 1 are independent random variables, each with with exponential distribution, mean $1/\lambda$. Flea 2 behaves similarly, but hops in the anticlockwise direction, the times between his hops having mean $1/\mu$. Show that the probability that they are at A at a given time $t > 0$ (starting from A at time $t = 0$) is

$$\frac{1}{3} + \frac{2}{3} \exp\left\{-\frac{3(\lambda+\mu)t}{2}\right\} \cos\left\{\frac{\sqrt{3}(\lambda-\mu)t}{2}\right\}.$$

2.8.2 Let $(X_t)_{t \geq 0}$ be a birth-and-death process with rates $\lambda_n = n\lambda$ and $\mu_n = n\mu$, and assume that $X_0 = 1$. Show that $h(t) = \mathbb{P}(X_t = 0)$ satisfies

$$h(t) = \int_0^t e^{-(\lambda+\mu)s}\{\mu + \lambda h(t-s)^2\} ds$$

and deduce that if $\lambda \neq \mu$ then

$$h(t) = (\mu e^{\mu t} - \mu e^{\lambda t})/(\mu e^{\mu t} - \lambda e^{\lambda t}).$$

2.9 Non-minimal chains

This book concentrates entirely on processes which are right-continuous and minimal. These are the simplest sorts of process and, overwhelmingly, the ones of greatest practical application. We have seen in this chapter that we can associate to each distribution λ and Q-matrix Q a unique such process, the Markov chain with initial distribution λ and generator matrix Q. Indeed we have taken the liberty of defining Markov chains to be those processes which arise in this way. However, these processes do not by any means exhaust the class of memoryless continuous-time processes with values in a countable set I. There are many more exotic possibilities, the general theory of which goes very much deeper than the account given in this book. It is in the nature of things that these exotic cases have received the greater attention among mathematicians. Here are some examples to help you imagine the possibilities.

Example 2.9.1

Consider a birth process $(X_t)_{t\geq 0}$ starting from 0 with rates $q_i = 2^i$ for $i \geq 0$. We have chosen these rates so that

$$\sum_{i=0}^{\infty} q_i^{-1} = \sum_{i=0}^{\infty} 2^{-i} < \infty$$

which shows that the process explodes (see Theorems 2.3.2 and 2.5.2). We have until now insisted that $X_t = \infty$ for all $t \geq \zeta$, where ζ is the explosion time. But another obvious possibility is to start the process off again from 0 at time ζ, and do the same for all subsequent explosions. An argument based on the memoryless property of the exponential distribution shows that for $0 \leq t_0 \leq \ldots \leq t_{n+1}$ this process satisfies

$$\mathbb{P}\big(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n\big) = p_{i_n i_{n+1}}(t_{n+1} - t_n)$$

for a semigroup of stochastic matrices $(P(t) : t \geq 0)$ on I. This is the defining property for a more general class of Markov chains. Note that the chain is no longer determined by λ and Q alone; the rule for bringing $(X_t)_{t\geq 0}$ back into I after explosion also has to be given.

Example 2.9.2

We make a variation on the preceding example. Suppose now that the jump chain of $(X_t)_{t\geq 0}$ is the Markov chain on $\mathbb{Z}$ which moves one step away from 0 with probability 2/3 and one step towards 0 with probability 1/3, and that $Y_0 = 0$. Let the transition rates for $(X_t)_{t\geq 0}$ be $q_i = 2^{|i|}$ for $i \in \mathbb{Z}$. Then $(X_t)_{t\geq 0}$ is again explosive. (A simple way to see this using some results of Chapter 3 is to check that $(Y_n)_{n\geq 0}$ is transient but $(X_t)_{t\geq 0}$ has an invariant distribution – by solution of the detailed balance equations. Then Theorem 3.5.3 makes explosion inevitable.) Now there are two ways in which $(X_t)_{t\geq 0}$ can explode, either $X_t \to -\infty$ or $X_t \to \infty$.

The process may again be restarted at 0 after explosion. Alternatively, we may choose the restart randomly, and according to the way that explosion occurred. For example

$$X_\zeta = \begin{cases} 0 & \text{if } X_t \to -\infty \text{ as } t \uparrow \zeta \\ Z & \text{if } X_t \to \infty \text{ as } t \uparrow \zeta \end{cases}$$

where Z takes values ± 1 with probability 1/2.