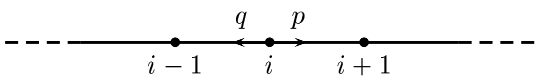
1.6 Recurrence and transience of random walks

In the last section we showed that recurrence was a class property, that all recurrent classes were closed and that all finite closed classes were recurrent. So the only chains for which the question of recurrence remains interesting are irreducible with infinite state-space. Here we shall study some simple and fundamental examples of this type, making use of the following criterion for recurrence from Theorem 1.5.3: a state i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

Example 1.6.1 (Simple random walk on Z)

The simple random walk on Z has diagram



where $0 < p = 1 - q < 1$. Suppose we start at 0. It is clear that we cannot return to 0 after an odd number of steps, so $p_{00}^{(2n+1)} = 0$ for all n. Any given sequence of steps of length $2n$ from 0 to 0 occurs with probability $p^n q^n$, there being n steps up and n steps down, and the number of such sequences is the number of ways of choosing the n steps up from $2n$. Thus

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n.$$

Stirling's formula provides a good approximation to $n!$ for large n: it is known that

$$n! \sim \sqrt{2\pi n}(n/e)^n \qquad \text{as } n \to \infty$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$. For a proof see W. Feller, *An Introduction to Probability Theory and its Applications, Vol I* (Wiley, New York, 3rd edition, 1968). At the end of this chapter we reproduce the argument used by Feller to show that

$$n! \sim A\sqrt{n}(n/e)^n \qquad \text{as } n \to \infty$$

for some $A \in [1, \infty)$. The additional work needed to show $A = \sqrt{2\pi}$ is omitted, as this fact is unnecessary to our applications.

For the n-step transition probabilities we obtain

$$p_{00}^{(2n)} = \frac{(2n)!}{(n!)^2}(pq)^n \sim \frac{(4pq)^n}{A\sqrt{n/2}} \qquad \text{as } n \to \infty.$$

In the symmetric case $p = q = 1/2$, so $4pq = 1$; then for some N and all $n \geq N$ we have

$$p_{00}^{(2n)} \geq \frac{1}{2A\sqrt{n}}$$

so

$$\sum_{n=N}^{\infty} p_{00}^{(2n)} \geq \frac{1}{2A} \sum_{n=N}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

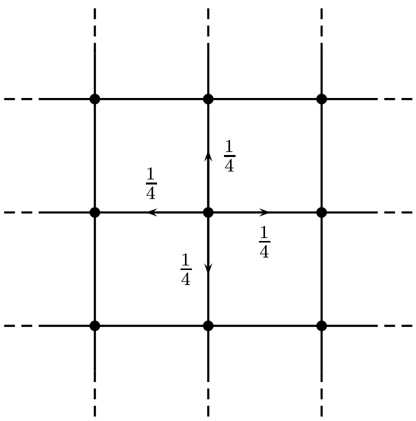
which shows that the random walk is recurrent. On the other hand, if $p \neq q$ then $4pq = r < 1$, so by a similar argument, for some N

$$\sum_{n=N}^{\infty} p_{00}^{(n)} \leq \frac{1}{A} \sum_{n=N}^{\infty} r^n < \infty$$

showing that the random walk is transient.

Example 1.6.2 (Simple symmetric random walk on Z^2)

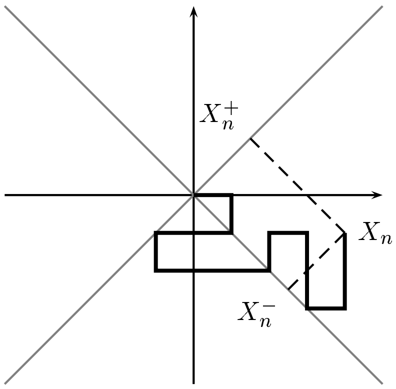
The simple symmetric random walk on Z^2 has diagram



and transition probabilities

$$p_{ij} = \begin{cases} 1/4 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose we start at 0. Let us call the walk X_n and write X_n^+ and X_n^- for the orthogonal projections of X_n on the diagonal lines $y = \pm x$:



Then X_n^+ and X_n^- are independent simple symmetric random walks on $2^{-1/2}Z$ and $X_n = 0$ if and only if $X_n^+ = 0 = X_n^-$. This makes it clear that for X_n we have

$$p_{00}^{(2n)} = \left(\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right)^2 \sim \frac{2}{A^2 n} \qquad \text{as } n \to \infty$$

by Stirling's formula. Then $\sum_{n=0}^{\infty} p_{00}^{(n)} = \infty$ by comparison with $\sum_{n=0}^{\infty} 1/n$ and the walk is recurrent.

Example 1.6.3 (Simple symmetric random walk on Z^3)

The transition probabilities of the simple symmetric random walk on Z^3 are given by

$$p_{ij} = \begin{cases} 1/6 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the chain jumps to each of its nearest neighbours with equal probability. Suppose we start at 0. We can only return to 0 after an even number $2n$ of steps. Of these $2n$ steps there must be i up, i down, j north, j south, k east and k west for some $i, j, k \geq 0$, with $i + j + k = n$. By counting the ways in which this can be done, we obtain

$$p_{00}^{(2n)} = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{(2n)!}{(i!j!k!)^2}\left(\frac{1}{6}\right)^{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \binom{n}{i\ j\ k}^2 \left(\frac{1}{3}\right)^{2n}.$$

Now

$$\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \binom{n}{i\ j\ k}\left(\frac{1}{3}\right)^n = 1$$

the left-hand side being the total probability of all the ways of placing n balls randomly into three boxes. For the case where $n = 3m$, we have

$$\binom{n}{i\ j\ k} = \frac{n!}{i!j!k!} \leq \binom{n}{m\ m\ m}$$

for all i, j, k, so

$$p_{00}^{(2n)} \leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\binom{n}{m\ m\ m}\left(\frac{1}{3}\right)^n \sim \frac{1}{2A^3}\left(\frac{6}{n}\right)^{3/2} \qquad \text{as } n \to \infty$$

by Stirling's formula. Hence, $\sum_{m=0}^{\infty} p_{00}^{(6m)} < \infty$ by comparison with $\sum_{n=0}^{\infty} n^{-3/2}$. But $p_{00}^{(6m)} \geq (1/6)^2\, p_{00}^{(6m-2)}$ and $p_{00}^{(6m)} \geq (1/6)^4\, p_{00}^{(6m-4)}$ for all m so we must have

$$\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty$$

and the walk is transient.

Exercises

1.6.1 The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

1.6.2 Show that the simple symmetric random walk in Z^4 is transient.