1.5 Recurrence and transience

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. We say that a state i is *recurrent* if

$$P_i(X_n = i \text{ for infinitely many } n) = 1.$$

We say that i is *transient* if

$$P_i(X_n = i \text{ for infinitely many } n) = 0.$$

Thus a recurrent state is one to which you keep coming back and a transient state is one which you eventually leave for ever. We shall show that every state is either recurrent or transient.

Recall that the *first passage time* to state i is the random variable T_i defined by

$$T_i(\omega) = \inf\{n \geq 1 : X_n(\omega) = i\}$$

where $\inf \emptyset = \infty$. We now define inductively the *rth passage time* $T_i^{(r)}$ to state i by

$$T_i^{(0)}(\omega) = 0, \quad T_i^{(1)}(\omega) = T_i(\omega)$$

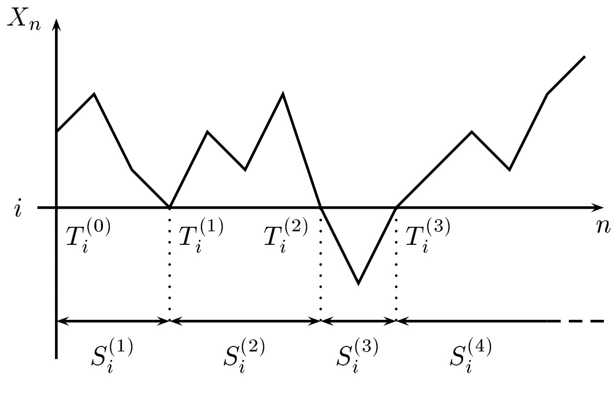
and, for $r = 0, 1, 2, \dots$,

$$T_i^{(r+1)}(\omega) = \inf\{n \geq T_i^{(r)}(\omega) + 1 : X_n(\omega) = i\}.$$

The length of the rth excursion to i is then

$$S_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)} & \text{if } T_i^{(r-1)} < \infty \\ 0 & \text{otherwise.} \end{cases}$$

The following diagram illustrates these definitions:



Our analysis of recurrence and transience will rest on finding the joint distribution of these excursion lengths.

Lemma 1.5.1. *For $r = 2, 3, \dots$, conditional on $T_i^{(r-1)} < \infty$, $S_i^{(r)}$ is independent of $\{X_m : m \le T_i^{(r-1)}\}$ and*

$$P(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = P_i(T_i = n).$$

Proof. Apply the strong Markov property at the stopping time $T = T_i^{(r-1)}$. It is automatic that $X_T = i$ on $T < \infty$. So, conditional on $T < \infty$, $(X_{T+n})_{n\ge 0}$ is Markov(δ_i, P) and independent of $X_0, X_1, \dots, X_T$. But

$$S_i^{(r)} = \inf\{n \ge 1 : X_{T+n} = i\},$$

so $S_i^{(r)}$ is the first passage time of $(X_{T+n})_{n\ge 0}$ to state i. □

Recall that the indicator function $1_{\{X_1=j\}}$ is the random variable equal to 1 if $X_1 = j$ and 0 otherwise. Let us introduce the *number of visits* V_i to i, which may be written in terms of indicator functions as

$$V_i = \sum_{n=0}^{\infty} 1_{\{X_n=i\}}$$

and note that

$$E_i(V_i) = E_i \sum_{n=0}^{\infty} 1_{\{X_n=i\}} = \sum_{n=0}^{\infty} E_i(1_{\{X_n=i\}}) = \sum_{n=0}^{\infty} P_i(X_n = i) = \sum_{n=0}^{\infty} p_{ii}^{(n)}.$$

Also, we can compute the distribution of V_i under P_i in terms of the *return probability*

$$f_i = P_i(T_i < \infty).$$

Lemma 1.5.2. *For $r = 0, 1, 2, \dots$, we have $P_i(V_i > r) = f_i^r$.*

Proof. Observe that if $X_0 = i$ then $\{V_i > r\} = \{T_i^{(r)} < \infty\}$. When $r = 0$ the result is true. Suppose inductively that it is true for r, then

$$\begin{aligned}
P_i(V_i > r+1) &= P_i(T_i^{(r+1)} < \infty) \\
&= P_i(T_i^{(r)} < \infty \text{ and } S_i^{(r+1)} < \infty) \\
&= P_i(S_i^{(r+1)} < \infty \mid T_i^{(r)} < \infty) P_i(T_i^{(r)} < \infty) \\
&= f_i f_i^r = f_i^{r+1}
\end{aligned}$$

by Lemma 1.5.1, so by induction the result is true for all r. □

Recall that one can compute the expectation of a non-negative integer-valued random variable as follows:

$$\sum_{r=0}^{\infty} P(V > r) = \sum_{r=0}^{\infty} \sum_{v=r+1}^{\infty} P(V = v) = \sum_{v=1}^{\infty} \sum_{r=0}^{v-1} P(V = v) = \sum_{v=1}^{\infty} v P(V = v) = E(V).$$

The next theorem is the means by which we establish recurrence or transience for a given state. Note that it provides two criteria for this, one in terms of the return probability, the other in terms of the n-step transition probabilities. Both are useful.

Theorem 1.5.3. *The following dichotomy holds:*

(i) *if $P_i(T_i < \infty) = 1$, then i is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$;*

(ii) *if $P_i(T_i < \infty) < 1$, then i is transient and $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$.*

In particular, every state is either transient or recurrent.

Proof. If $P_i(T_i < \infty) = 1$, then, by Lemma 1.5.2,

$$P_i(V_i = \infty) = \lim_{r\to\infty} P_i(V_i > r) = 1$$

so i is recurrent and

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = E_i(V_i) = \infty.$$

On the other hand, if $f_i = P_i(T_i < \infty) < 1$, then by Lemma 1.5.2

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = E_i(V_i) = \sum_{r=0}^{\infty} P_i(V_i > r) = \sum_{r=0}^{\infty} f_i^r = \frac{1}{1-f_i} < \infty$$

so $P_i(V_i = \infty) = 0$ and i is transient. $\square$

From this theorem we can go on to solve completely the problem of recurrence or transience for Markov chains with finite state-space. Some cases of infinite state-space are dealt with in the following chapter. First we show that recurrence and transience are *class properties*.

Theorem 1.5.4. *Let C be a communicating class. Then either all states in C are transient or all are recurrent.*

Proof. Take any pair of states $i, j \in C$ and suppose that i is transient. There exist $n, m \geq 0$ with $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$, and, for all $r \geq 0$

$$p_{ii}^{(n+r+m)} \geq p_{ij}^{(n)} p_{jj}^{(r)} p_{ji}^{(m)}$$

so

$$\sum_{r=0}^{\infty} p_{jj}^{(r)} \leq \frac{1}{p_{ij}^{(n)} p_{ji}^{(m)}} \sum_{r=0}^{\infty} p_{ii}^{(n+r+m)} < \infty$$

by Theorem 1.5.3. Hence j is also transient by Theorem 1.5.3. $\square$

In the light of this theorem it is natural to speak of a recurrent or transient class.

Theorem 1.5.5. *Every recurrent class is closed.*

Proof. Let C be a class which is not closed. Then there exist $i \in C$, $j \notin C$ and $m \geq 1$ with

$$P_i(X_m = j) > 0.$$

Since we have

$$P_i(\{X_m = j\} \cap \{X_n = i \text{ for infinitely many } n\}) = 0$$

this implies that

$$P_i(X_n = i \text{ for infinitely many } n) < 1$$

so i is not recurrent, and so neither is C. $\square$

Theorem 1.5.6. *Every finite closed class is recurrent.*

Proof. Suppose C is closed and finite and that $(X_n)_{n\geq 0}$ starts in C. Then for some $i \in C$ we have

$$\begin{aligned} 0 &< \mathbb{P}(X_n = i \text{ for infinitely many } n) \\ &= \mathbb{P}(X_n = i \text{ for some } n)\mathbb{P}_i(X_n = i \text{ for infinitely many } n) \end{aligned}$$

by the strong Markov property. This shows that i is not transient, so C is recurrent by Theorems 1.5.3 and 1.5.4. $\square$

It is easy to spot closed classes, so the transience or recurrence of finite classes is easy to determine. For example, the only recurrent class in Example 1.2.2 is $\{5, 6\}$, the others being transient. On the other hand, infinite closed classes may be transient: see Examples 1.3.3 and 1.6.3.

We shall need the following result in Section 1.8. Remember that irreducibility means that the chain can get from any state to any other, with positive probability.

Theorem 1.5.7. *Suppose P is irreducible and recurrent. Then for all $j \in I$ we have $P(T_j < \infty) = 1$.*

Proof. By the Markov property we have

$$P(T_j < \infty) = \sum_{i \in I} P(X_0 = i)P_i(T_j < \infty)$$

so it suffices to show $P_i(T_j < \infty) = 1$ for all $i \in I$. Choose m with $p_{ji}^{(m)} > 0$. By Theorem 1.5.3, we have

$$\begin{aligned} 1 &= P_j(X_n = j \text{ for infinitely many } n) \\ &= P_j(X_n = j \text{ for some } n \geq m+1) \\ &= \sum_{k \in I} P_j(X_n = j \text{ for some } n \geq m+1 \mid X_m = k)P_j(X_m = k) \\ &= \sum_{k \in I} P_k(T_j < \infty)p_{jk}^{(m)} \end{aligned}$$

where the final equality uses the Markov property. But $\sum_{k \in I} p_{jk}^{(m)} = 1$ so we must have $P_i(T_j < \infty) = 1$. $\square$

Exercises

1.5.1 In Exercise 1.2.1, which states are recurrent and which are transient?

1.5.2 Show that, for the Markov chain $(X_n)_{n\geq 0}$ in Exercise 1.3.4 we have

$$P(X_n \to \infty \text{ as } n \to \infty) = 1\,.$$

Suppose, instead, the transition probabilities satisfy

$$p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}\,.$$

For each $\alpha \in (0, \infty)$ find the value of $P(X_n \to \infty \text{ as } n \to \infty)$.

1.5.3 (First passage decomposition). Denote by T_j the first passage time to state j and set

$$f_{ij}^{(n)} = P_i(T_j = n).$$

Justify the identity

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \qquad \text{for } n \geq 1$$

and deduce that

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s) P_{ij}(s)$$

where

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n, \quad F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n.$$

Hence show that $P_i(T_i < \infty) = 1$ if and only if

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$$

without using Theorem 1.5.3.

1.5.4 A random sequence of non-negative integers $(F_n)_{n \geq 0}$ is obtained by setting $F_0 = 0$ and $F_1 = 1$ and, once $F_0, \ldots, F_n$ are known, taking F_{n+1} to be either the sum or the difference of F_{n-1} and F_n, each with probability $1/2$. Is $(F_n)_{n \geq 0}$ a Markov chain?

By considering the Markov chain $X_n = (F_{n-1}, F_n)$, find the probability that $(F_n)_{n \geq 0}$ reaches 3 before first returning to 0.

Draw enough of the flow diagram for $(X_n)_{n \geq 0}$ to establish a general pattern. Hence, using the strong Markov property, show that the hitting probability for $(1, 1)$, starting from $(1, 2)$, is $(3 - \sqrt{5})/2$.

Deduce that $(X_n)_{n \geq 0}$ is transient. Show that, moreover, with probability 1, $F_n \to \infty$ as $n \to \infty$.