## 1.11 Appendix: recurrence relations

Recurrence relations often arise in the linear equations associated to Markov chains. Here is an account of the simplest cases. A more specialized case was dealt with in Example 1.3.4. In Example 1.1.4 we found a recurrence relation of the form

$$x_{n+1} = ax_n + b.$$

We look first for a constant solution $x_n = x$; then $x = ax + b$, so provided $a \neq 1$ we must have $x = b/(1-a)$. Now $y_n = x_n - b/(1-a)$ satisfies $y_{n+1} = ay_n$, so $y_n = a^n y_0$. Thus the general solution when $a \neq 1$ is given by

$$x_n = Aa^n + b/(1-a)$$

where $A$ is a constant. When $a = 1$ the general solution is obviously

$$x_n = x_0 + nb.$$

In Example 1.3.3 we found a recurrence relation of the form

$$ax_{n+1} + bx_n + cx_{n-1} = 0$$

where $a$ and $c$ were both non-zero. Let us try a solution of the form $x_n = \lambda^n$; then $a\lambda^2 + b\lambda + c = 0$. Denote by $\alpha$ and $\beta$ the roots of this quadratic. Then

$$y_n = A\alpha^n + B\beta^n$$

is a solution. If $\alpha \neq \beta$ then we can solve the equations

$$x_0 = A + B, \quad x_1 = A\alpha + B\beta$$

so that $y_0 = x_0$ and $y_1 = x_1$; but

$$a(y_{n+1} - x_{n+1}) + b(y_n - x_n) + c(y_{n-1} - x_{n-1}) = 0$$

for all $n$, so by induction $y_n = x_n$ for all $n$. If $\alpha = \beta \neq 0$, then

$$y_n = (A + nB)\alpha^n$$

is a solution and we can solve

$$x_0 = A\alpha^n, \quad x_1 = (A + B)\alpha^n$$

so that $y_0 = x_0$ and $y_1 = x_1$; then, by the same argument, $y_n = x_n$ for all $n$. The case $\alpha = \beta = 0$ does not arise. Hence the general solution is given by

$$x_n = \begin{cases} A\alpha^n + B\beta^n & \text{if } \alpha \neq \beta \\ (A + nB)\alpha^n & \text{if } \alpha = \beta. \end{cases}$$

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### 1.12 Appendix: asymptotics for $n!$

Our analysis of recurrence and transience for random walks in Section 1.6 rested heavily on the use of the asymptotic relation

$$n! \sim A\sqrt{n}(n/e)^n \quad \text{as } n \to \infty$$

for some $A \in [1, \infty)$. Here is a derivation.

We make use of the power series expansions for $|t| < 1$

$$\log(1+t) = t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \dots$$
$$\log(1-t) = -t - \tfrac{1}{2}t^2 - \tfrac{1}{3}t^3 - \dots .$$

By subtraction we obtain

$$\tfrac{1}{2}\log\left(\frac{1+t}{1-t}\right) = t + \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + \dots .$$

Set $A_n = n!/(n^{n+1/2}e^{-n})$ and $a_n = \log A_n$. Then, by a straightforward calculation

$$a_n - a_{n+1} = (2n+1)\frac{1}{2}\log\left(\frac{1+(2n+1)^{-1}}{1-(2n+1)^{-1}}\right) - 1.$$

By the series expansion written above we have

$$\begin{aligned}
a_n - a_{n+1} &= (2n+1)\left\{\frac{1}{(2n+1)} + \frac{1}{3}\frac{1}{(2n+1)^3} + \frac{1}{5}\frac{1}{(2n+1)^5} + \dots\right\} - 1 \\
&= \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \dots \\
&\le \frac{1}{3}\left\{\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots\right\} \\
&= \frac{1}{3}\frac{1}{(2n+1)^2 - 1} = \frac{1}{12n} - \frac{1}{12(n+1)} .
\end{aligned}$$

It follows that $a_n$ decreases and $a_n - 1/(12n)$ increases as $n \to \infty$. Hence $a_n \to a$ for some $a \in [0, \infty)$ and hence $A_n \to A$, as $n \to \infty$, where $A = e^a$.