1

Introduction

This book is about a certain sort of random process. The characteristic property of this sort of process is that it retains *no memory* of where it has been in the past. This means that only the current state of the process can influence where it goes next. Such a process is called a *Markov process*. We shall be concerned exclusively with the case where the process can assume only a finite or countable set of states, when it is usual to refer it as a *Markov chain*.

Examples of Markov chains abound, as you will see throughout the book. What makes them important is that not only do Markov chains model many phenomena of interest, but also the lack of memory property makes it possible to predict how a Markov chain may behave, and to compute probabilities and expected values which quantify that behaviour. In this book we shall present general techniques for the analysis of Markov chains, together with many examples and applications. In this introduction we shall discuss a few very simple examples and preview some of the questions which the general theory will answer.

We shall consider chains both in *discrete time*

$$n \in Z^+ = \{0, 1, 2, \dots\}$$

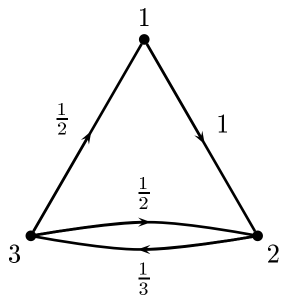
and *continuous time*

$$t \in R^+ = [0, \infty).$$

The letters n, m, k will always denote integers, whereas t and s will refer to real numbers. Thus we write $(X_n)_{n \geq 0}$ for a discrete-time process and $(X_t)_{t \geq 0}$ for a continuous-time process.

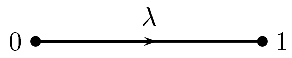
Markov chains are often best described by diagrams, of which we now give some simple examples:

(i) (*Discrete time*)



You move from state 1 to state 2 with probability 1. From state 3 you move either to 1 or to 2 with equal probability 1/2, and from 2 you jump to 3 with probability 1/3, otherwise stay at 2. We might have drawn a loop from 2 to itself with label 2/3. But since the total probability on jumping from 2 must equal 1, this does not convey any more information and we prefer to leave the loops out.

(ii) (*Continuous time*)

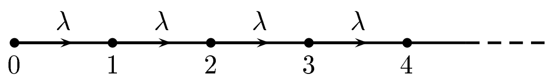


When in state 0 you wait for a random time with exponential distribution of parameter $\lambda \in (0, \infty)$, then jump to 1. Thus the density function of the waiting time T is given by

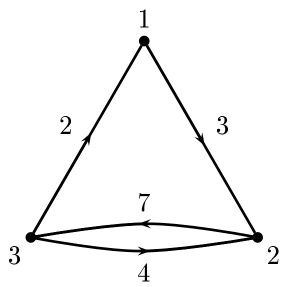
$$f_T(t) = \lambda e^{-\lambda t} \qquad \text{for } t \geq 0.$$

We write $T \sim E(\lambda)$ for short.

(iii) (*Continuous time*)

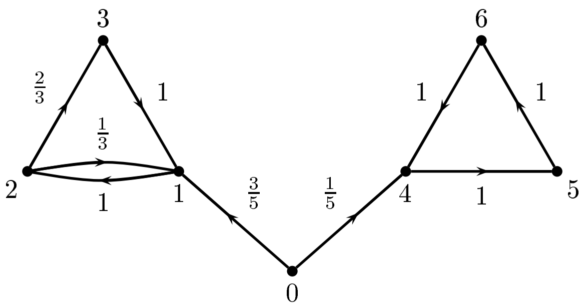


Here, when you get to 1 you do not stop but after another independent exponential time of parameter λ jump to 2, and so on. The resulting process is called the *Poisson process of rate* λ.



(iv) (*Continuous time*)
In state 3 you take two independent exponential times $T_1 \sim E(2)$ and $T_2 \sim E(4)$; if T_1 is the smaller you go to 1 after time T_1, and if T_2 is the smaller you go to 2 after time T_2. The rules for states 1 and 2 are as given in examples (ii) and (iii). It is a simple matter to show that the time spent in 3 is exponential of parameter $2+4=6$, and that the probability of jumping from 3 to 1 is $2/(2+4)=1/3$. The details are given later.

(v) (*Discrete time*)



We use this example to anticipate some of the ideas discussed in detail in Chapter 1. The states may be partitioned into *communicating classes*, namely $\{0\}$, $\{1,2,3\}$ and $\{4,5,6\}$. Two of these classes are *closed*, meaning that you cannot escape. The closed classes here are *recurrent*, meaning that you return again and again to every state. The class $\{0\}$ is *transient*. The class $\{4,5,6\}$ is *periodic*, but $\{1,2,3\}$ is not. We shall show how to establish the following facts by solving some simple linear equations. You might like to try from first principles.

(a) Starting from 0, the probability of hitting 6 is $1/4$.
(b) Starting from 1, the probability of hitting 3 is 1.
(c) Starting from 1, it takes on average three steps to hit 3.
(d) Starting from 1, the long-run proportion of time spent in 2 is $3/8$.

Let us write p_{ij}^n for the probability starting from i of being in state j after n steps. Then we have:

(e) $\lim_{n\to\infty} p_{01}^n = 9/32$;
(f) p_{04}^n does not converge as $n \to \infty$;
(g) $\lim_{n\to\infty} p_{04}^{3n} = 1/124$.