Advanced Probability 4

7.9 Let μ denote Wiener measure on $W = \{x \in C([0,1],\mathbb{R}) : x_0 = 0\}$. For $a \in \mathbb{R}$, define a new probability measure μ_a on W by

$$d\mu_a/d\mu(x) = \exp(ax_1 - a^2/2).$$

Show that under μ_a the coordinate process remains Gaussian, and identify its distribution. Deduce that $\mu(A) > 0$ for every non-empty open set $A \subseteq W$.

7.10 Let $X = (X_t)_{0\leq t\leq 1}$ be a Brownian motion, starting from 0. Denote by μ the law of B on $W = C([0,1],\mathbb{R})$. For each $y \in \mathbb{R}$, set

$$Z_t^y = yt + (X_t - tX_1)$$

and denote by μ^y the law of $Z^y = (Z_t^y)_{0\leq t\leq 1}$ on W. Show that, for any bounded measurable function $F : W \to \mathbb{R}$ and for $f(y) = \mu^y(F)$ we have, almost surely,

$$\mathbb{E}(F(X)|X_1) = f(X_1).$$

7.11 Let D be a bounded open set in $\mathbb{R}^n$ and let $h : \bar{D} \to \mathbb{R}$ be a bounded continuous function, harmonic in D. Show that, for all $x \in D$,

$$\inf_{y\in\partial D} h(y) \leq h(x) \leq \sup_{y\in\partial D} h(y).$$

7.12 (i) Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^2$, starting from (x,y). Compute the distribution of X_T, where

$$T = \inf\{t \geq 0 : X_t \notin H\}$$

and where H is the upper half plane $\{(x,y) : y > 0\}$.

(ii) Show that, for any bounded continuous function $u : \bar{H} \to \mathbb{R}$, harmonic in H, with $u(x,0) = f(x)$ for all $x \in \mathbb{R}$, we have

$$u(x,y) = \int_{\mathbb{R}} f(s)\frac{1}{\pi}\,\frac{y}{(x-s)^2+y^2}\,ds.$$

(iii) Let D be any open set in $\mathbb{R}^2$ for which there exists a continuous homeomorphism $g : \bar{H} \to \bar{D}$, which is conformal in H. Show that, if u is harmonic in D, then $u \circ g$ is harmonic in H.

(iv) Find an explicit integral representation for bounded continuous functions $u : \bar{D} \to \mathbb{R}$, harmonic in D, in terms of their values on the boundary of D.

(v) Determine the exit distribution of Brownian motion from D.

8.1 Let $(X_k : k \in \mathbb{N})$ be a sequence of independent Poisson random variables, X_k having parameter $\lambda_k \in [0,\infty]$ for all k. Set $X = \sum_k X_k$ and $\lambda = \sum_k \lambda_k$. Show that X is a Poisson random variable of parameter λ.

8.2 Let $(Y_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed, $\mathbb{N}$-valued random variables. Let X be a Poisson random variable of parameter $\lambda \in [0, \infty]$. Define for $k \in \mathbb{N}$

$$X_k = \sum_{n=1}^{X} 1_{\{Y_n = k\}}, \quad \lambda_k = \lambda p_k$$

where $p_k = \mathbb{P}(Y_1 = k)$ and we agree that $\infty.0 = 0$. Show that the random variables $(X_k : k \in \mathbb{N})$ are independent and X_k has Poisson distribution of parameter λ_k for all k.

8.3 Let $(E, \mathcal{E}, K)$ be a finite measure space and let $g \in L^1(K)$. Let $\tilde{M} = M - \mu$ be a compensated Poisson random measure on $(0, \infty) \times E$, with intensity $\mu = dt \otimes K$. Set

$$\tilde{M}_t(g) = \begin{cases} \int_{(0,t] \times E} g(y) \tilde{M}(ds, dy), & \text{if } M((0,t] \times E) < \infty \text{ for all } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $(\tilde{M}_t(g))_{t \geq 0}$ is a cadlag martingale with stationary independent increments. Show further that, for all $t \geq 0$,

$$\mathbb{E}(\tilde{M}_t^2(g)) = t \int_E g(y)^2 K(dy)$$

and

$$\mathbb{E}(e^{iu\tilde{M}_t(g)}) = \exp\left\{ t \int_E (e^{iug(y)} - 1 - iug(y)) K(dy) \right\}.$$

9.1 Let $(X_t)_{t \geq 0}$ be a Lévy process with characteristic exponent ψ. Show that, for all $u \in \mathbb{R}$, the following process is a martingale

$$M_t^u = \exp\{iuX_t - t\psi(u)\}.$$

9.2 Say that a Lévy process $(X_t)_{t \geq 0}$ satisfies the scaling relation with exponent $\alpha \in (0, \infty)$ if

$$(cX_{c^{-\alpha}t})_{t \geq 0} \sim (X_t)_{t \geq 0}, \quad c \in (0, \infty).$$

For example, Brownian motion satisfies the scaling relation with exponent 2. Find, for each $\alpha \in (0, 2)$, a Lévy process having a scaling relation with exponent α.

9.3 Let $(X_t)_{t \geq 0}$ be the Lévy process corresponding to the Lévy triple (a, b, K). Show that, if K consists of finitely many atoms, then $(X_t)_{t \geq 0}$ can be written as a linear combination of a Brownian motion, a uniform drift and finitely many Poisson processes.