Advanced Probability 1

1.1 Let $X, Y \in L^1(\mathbb{P})$ and let $\mathcal{G}$ be a σ-algebra. Show that

$$\mathbb{E}(X+Y|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Y|\mathcal{G}) \quad \text{almost surely.}$$

Show also that, if Y is $\mathcal{G}$-measurable and $\mathbb{E}(X1_A) \le \mathbb{E}(Y1_A)$ for all $A \in \mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) \le Y$ almost surely.

1.2 Show that, for any sequence of non-negative random variables $(X_n : n \in \mathbb{N})$ and any σ-algebra $\mathcal{G}$,

$$\mathbb{E}(\liminf X_n|\mathcal{G}) \le \liminf \mathbb{E}(X_n|\mathcal{G}) \quad \text{almost surely.}$$

1.3 Let X and Y be random variables and let $\lambda \in (0,\infty)$. Show that, if X and $Y - X$ are independent exponential random variables of parameter λ, then Y has density $\lambda^2 y e^{-\lambda y}$ on $(0,\infty)$ and $\mathbb{P}(X \le x|Y) = (x/Y) \wedge 1$ almost surely, for all $x \ge 0$.

1.4 Let X and Y be integrable random variables such that, almost surely,

$$\mathbb{E}(X|Y) = Y \quad \text{and} \quad \mathbb{E}(Y|X) = X.$$

Show that $X = Y$ almost surely. *You may find it helpful to consider the quantity*

$$\mathbb{E}((X-Y)1_{\{X\le c, Y>c\}}) + \mathbb{E}((X-Y)1_{\{X\le c, Y\le c\}}).$$

1.5 Let X be a non-negative random variable and let Y be a version of $\mathbb{E}(X|\mathcal{G})$. Show that $\{X > 0\} \subseteq \{Y > 0\}$ almost surely, that is, $1_{\{X>0\}} \le 1_{\{Y>0\}}$ almost surely. Show further that, for all $A \in \mathcal{G}$, if $\{X > 0\} \subseteq A$ almost surely then $\{Y > 0\} \subseteq A$ almost surely.

2.1 Let $(Y_n : n \in \mathbb{N})$ be a sequence of independent integrable random variables of mean 0. Set $X_0 = 0$ and $X_n = Y_1 + \cdots + Y_n$ for $n \ge 1$. Show that $(X_n)_{n\ge0}$ is a martingale.

2.2 Let $(Z_n : n \in \mathbb{N})$ be a sequence of independent random variables, such that $Z_n \ge 0$ and $\mathbb{E}(Z_n) = 1$ for all n. Set $X_0 = 1$ and $X_n = \prod_{k=1}^n Z_k$ for $n \ge 1$. Show that $(X_n)_{n\ge0}$ is a martingale.

2.3 Let $(X_n)_{n\ge0}$ be an integrable process, taking values in a countable set $S \subseteq \mathbb{R}$. Show that $(X_n)_{n\ge0}$ is a martingale in its natural filtration if and only if, for all n and for all $x_0, \dots, x_n \in S$, whenever the conditioning event has positive probability, we have

$$\mathbb{E}(X_{n+1}|X_0 = x_0, \dots, X_n = x_n) = x_n.$$

2.4 Let $(X_n)_{n\ge0}$ be a martingale and let f be a convex function on $\mathbb{R}$ such that $f(X_n)$ is integrable for all n. Show that $(f(X_n))_{n\ge0}$ is a submartingale.

2.5 Let S and T be stopping times and let $X = (X_n)_{n\ge0}$ be an integrable process. Recall that we define the stopped process X^T by $X^T_n = X_{T\wedge n}$ and we set

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \le n\} \in \mathcal{F}_n \text{ for all } n \ge 0\}.$$

Show that $S \wedge T$ is a stopping time. Show also that $\mathcal{F}_T$ is a σ-algebra and that $\mathcal{F}_S \subseteq \mathcal{F}_T$ if $S \leq T$. Show that $X_T 1_{\{T<\infty\}}$ is an $\mathcal{F}_T$-measurable random variable. Show finally that X^T is an adapted process, and that X^T is an integrable process whenever X is an integrable process.

2.6 Let $(X_n)_{n\geq 0}$ be a martingale and let T be a stopping time. Show that, under each one of the following conditions, X_T is integrable and $\mathbb{E}(X_T) = \mathbb{E}(X_0)$:

(a) $T \leq n$ for some $n \geq 0$,
(b) there is a constant $C < \infty$ such that $|X_n| \leq C$ for all $n \leq T$ almost surely,
(c) $\mathbb{E}(T) < \infty$ and there is a constant $C < \infty$ such that $|X_{n+1} - X_n| \leq C$ for all $n < T$ almost surely.

2.7 Let $X \in L^2(\mathbb{P})$ and set $X_n = \mathbb{E}(X|\mathcal{F}_n)$, where $(\mathcal{F}_n)_{n\geq 0}$ is a given filtration. Show that, for all $m \leq n$,

$$\|X_m\|_2^2 + \|X_m - X_n\|_2^2 = \|X_n\|_2^2.$$

Hence show there exists $Y \in L^2(\mathbb{P})$ such that $X_n \to Y$ in L^2. Show further that $Y = X$ almost surely if and only if X is $\mathcal{F}_\infty$-measurable, where $\mathcal{F}_\infty = \sigma(\mathcal{F}_n : n \geq 0)$.

2.8 Let $(X_n)_{n\geq 0}$ be a martingale, starting from 0. Show that $(X_n)_{n\geq 0}$ is bounded in L^2 if and only if $\sum_n \|X_{n+1} - X_n\|_2^2 < \infty$.

2.9 Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$ for all n. Show that the series $\sum_n X_n/n$ converges almost surely.

2.10 Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = -1/p_n) = p_n$ and $\mathbb{P}(X_n = 1/q_n) = q_n$, where $p_n = 1/n^2$ and $p_n + q_n = 1$. Set $S_n = X_1 + \cdots + X_n$. Show that S_n/n converges almost surely as $n \to \infty$. Show also that $(S_n)_{n\geq 0}$ is a martingale but $S_n \to \infty$ almost surely as $n \to \infty$.

3.1 *Pólya's urn.* At time 0, an urn contains two balls, one black, the other white. Suppose we repeatedly choose a ball at random from the urn and replace it together with a new ball of the same colour. Then, after n steps, there are $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of steps in which a black ball was chosen. Let $M_n = (B_n + 1)/(n + 2)$ the proportion of black balls in the urn after n steps. Show that $(M_n)_{n\geq 0}$ is a martingale, relative to a filtration which you should specify. Show also that

$$\mathbb{P}(B_n = k) = (n + 1)^{-1}, \quad k = 0, 1, \ldots, n.$$

Deduce that there is a random variable Θ such that $M_n \to \Theta$ almost surely and find the distribution of Θ. For $\theta \in [0, 1]$, set

$$N_n^\theta = \frac{(n + 1)!}{B_n!(n - B_n)!}\, \theta^{B_n}(1 - \theta)^{n-B_n}.$$

Show that $(N_n^\theta)_{n\geq 0}$ is a martingale.

3.2 *Bayes' urn.* A random number Θ is chosen uniformly in $[0, 1]$, and a coin with probability Θ of heads is minted. The coin is tossed repeatedly. Let B_n be the number of heads in n tosses. Show that the process $(B_n)_{n\geq 0}$ has the same distribution as the process $(B_n)_{n\geq 0}$ in Example 3.1. Show that N_n^θ is a conditional density function of Θ given $B_1, \ldots, B_n$.