RANDOM PLANAR GEOMETRY

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Preface

These lecture notes are for the University of Cambridge Part III course Random Planar Geometry, given Lent 2020. Please notify jpmiller@statslab.cam.ac.uk for corrections.

1. Introduction

In the course, we will cover three main topics:

- Random planar trees
- Random planar maps
- Schramm-Loewner evolution

1.1. **Random planar trees.** A tree is a connected graph without cycles. A plane tree is a tree with an ordering on the vertices that tells you how to draw it in the plane. If we fix the number of edges to be some positive integer k, then there are only a finite number of plane trees and therefore one can pick one uniformly at random. This is an example of a random plane tree. Random plane trees arise in many contexts in probability. In this course, we will discuss properties of random plane trees and the continuous object which describes their scaling limit (the continuum random tree).

1.2. **Random planar maps.** A planar map is a graph together with an embedding into the plane so that no two edges cross. The faces of a planar map are the connected components of the complement of its edges. For example, one can consider triangulations (each face has three adjacent edges) or quadrangulations (each face has four adjacent edges). In this course, we will focus on quadrangulations since certain aspects of their analysis is simpler than in the case of other types of planar maps. There are only a finite number of quadrangulations with a fixed positive integer k number of faces and therefore one can choose one uniformly at random. This is an example of a random planar map. In this course, we will discuss properties of random planar maps and the continuous object which describes their scaling limit (the Brownian map).

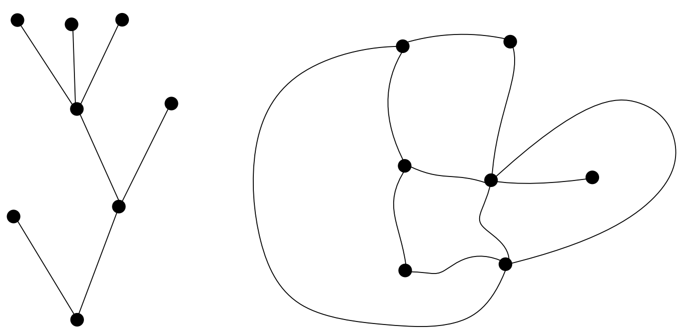

FIGURE 1.1. **Left:** A planar tree with 8 vertices (and 7 edges). **Right:** A (planar) quadrangulation. Note that the unbounded face also has four adjacent edges.

1.3. **Schramm-Loewner evolution.** The Schramm-Loewner evolution (SLE) is a random fractal curve which lives in a domain D in the complex plane $\mathbf{C}$. It was introduced by Schramm in 1999 to describe the scaling limits of interfaces in two-dimensional discrete models from statistical mechanics. It has been a transformative idea which has led to new unexpected links between a number of probabilistic models and also other areas of mathematics.

Here are three important examples where SLE's arise.

Example 1.1 (Loop-erased random walk on $\mathbf{Z}^2$)**.** A (simple) random walk on $\mathbf{Z}^2$ is a particle X_n which in each time step goes up/down/left/right with equal probability 1/4. The loop-erasure of X_n is defined by erasing the loops that X_n makes chronologically. It is an important object in probability because it is connected to many other probabilistic models (e.g., uniform spanning trees, dimers, sand-pile models). See the left side of Figure 1.2 for a simulation of a long random walk

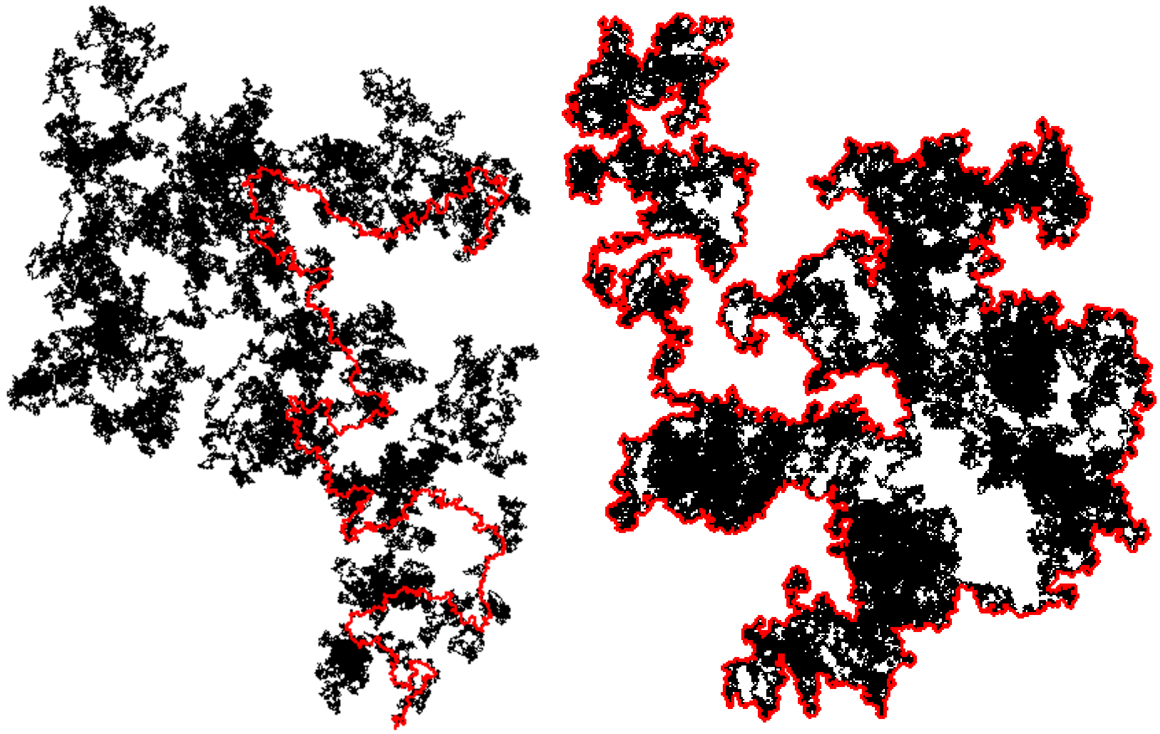

FIGURE 1.2. **Left:** A random walk (black) on $\mathbf{Z}^2$ and its loop-erasure (red). It was proved by Lawler-Schramm-Werner that the scaling limit of the loop-erasure is given by an SLE_2 curve. **Right:** The range of a planar Brownian motion shown in black and its outer boundary shown in red. It was conjectured by Mandelbrot that the dimension of the outer boundary is equal to $\frac{4}{3}$. Mandelbrot's conjecture was proved by Lawler-Schramm-Werner using SLE.

together with its loop-erasure. By Donsker's invariance principle, $X_{\lfloor nt \rfloor}/\sqrt{n}$ converges in the limit to a two-dimensional Brownian motion. A natural question to ask is what continuous object describes the scaling limit of the loop-erasure of X_n. It was proved by Lawler-Schramm-Werner that it is given by an SLE_2 curve.

Example 1.2 (Outer boundary of Brownian motion)**.** Suppose that $X = (B_1, B_2)$ is a planar Brownian motion. That is, B_1, B_2 are independent standard Brownian motions. The *outer boundary* of $X([0,1])$ is the boundary of the unbounded component of $\mathbf{C} \setminus X([0,1])$. See the right side of Figure 1.2 for a simulation of a planar Brownian motion with emphasis on its outer boundary. Mandelbrot conjectured state that the Hausdorff dimension, a measure theoretic notion of dimension, is equal to $\frac{4}{3}$. This conjecture was proved by Lawler-Schramm-Werner.

Example 1.3 (Percolation interface)**.** Consider the hexagonal lattice in the plane. We color each hexagon either "white" or "black" independently with equal probability $\frac{1}{2}$. See Figure 1.3 for a numerical simulation. A famous question in probability for many years was to describe the large scale behavior of the interfaces between the white and the black sites. This problem was solved by Smirnov, who showed that they converge in the limit to SLE_6 curves.

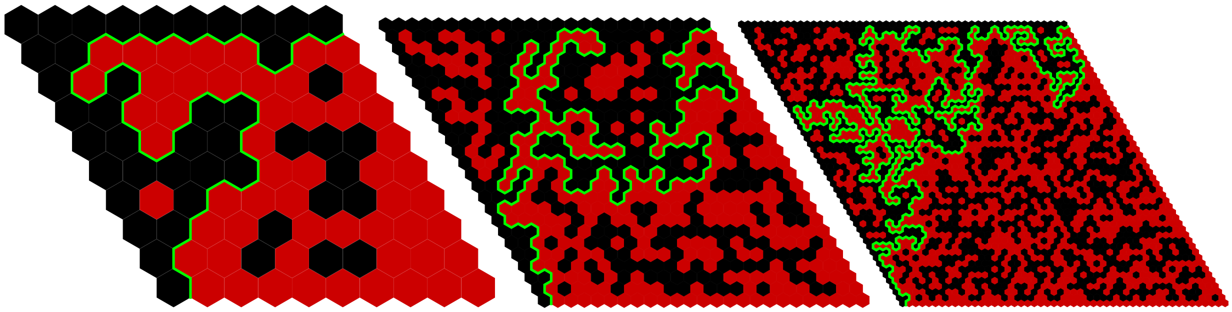

FIGURE 1.3. Critical percolation on a lozenge shaped subset of the hexagonal lattice in $\mathbf{C}$ with black boundary conditions on the left and top sides and red boundary conditions on the bottom and right sides. This choice of boundary conditions forces the existence of a unique interface (green) from the bottom corner of the lozenge to the top which has black (resp. red) hexagons on its left (resp. right) side. It was proved by Smirnov that the scaling limit of this interface converges in the limit to an SLE_6 curve. The left, middle, and right lozenges respectively have side length 10, 25, and 50.

Famous open question: prove the same thing for any *other* planar lattice, such as $\mathbf{Z}^2$.

2. Plane trees

Throughout, we will let $\mathbf{N} = \{1, 2, \ldots\}$ denote the positive integers and we will take the convention that $\mathbf{N}^0 = \{\emptyset\}$. We let $\mathcal{U} = \cup_{n=0}^{\infty} \mathbf{N}^n$. Then an element of $\mathcal{U}$ is a finite sequence $(u^1, \ldots, u^n)$ of elements of $\mathbf{N}$. We let $|u| = n$ be the number of elements in the sequence $u \in \mathcal{U}$. If $u = (u^1, \ldots, u^k)$ and $v = (v^1, \ldots, v^\ell)$ are elements of $\mathcal{U}$, then we write $uv = (u^1, \ldots, u^k, v^1, \ldots, v^\ell)$ and we take the convention that $u\phi = \phi u = u$. We let $\pi \colon \mathcal{U} \setminus \{\emptyset\} \to \mathcal{U}$ be defined by setting $\pi((u^1, \ldots, u^n)) = (u^1, \ldots, u^{n-1})$. A *plane tree* τ is a finite subset of $\mathcal{U}$ which satisfies the following properties:

(i) $\phi \in \tau$ (Root)
(ii) For all $u \in \tau \setminus \{\emptyset\}$ we have that $\pi(u) \in \tau$ (Parent relation)
(iii) For all $u \in \tau$, there exists a non-negative integer $k_u(\tau)$ so that for all $j \in \mathbf{N}$ we have that $uj \in \tau$ if and only if $1 \le j \le k_u(\tau)$. (Children)

We let $\mathbf{T}$ be the set of plane trees. For each $\tau \in \mathbf{T}$, we let $|\tau|$ be the number of edges of τ which we note is equal to one less than the number of vertices in τ. For each $k \geq 0$ we let $\mathbf{T}_k = \{\tau \in \mathbf{T} : |\tau| = k\}$. One basic fact (Example Sheet 1) is that

$$|\mathbf{T}_k| = \frac{1}{k+1}\binom{2k}{k}.$$

It is natural to encode a plane tree in terms of its *contour function* (or Dyck path). The contour function of $\tau \in \mathbf{T}$ is defined as follows.

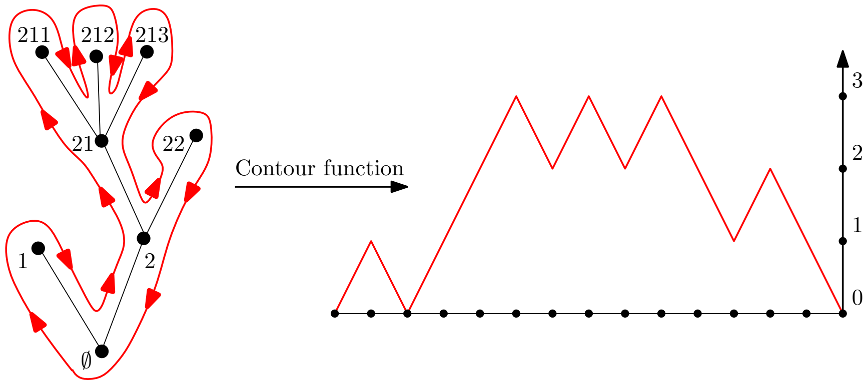

FIGURE 2.1. A plane tree with 8 vertices (and 7 edges) together with its contour function.

- We suppose that τ has been embedded into $\mathbf{T}$ so that its edges do not cross and each edge is a straight line with Euclidean length 1.
- We then start a particle at the root ($\emptyset$) which travels at unit speed along the edges of τ in a depth first manner. Note that the time required for the particle to visit every vertex of the tree and then return to the root is equal to $2|\tau|$ since every edge of τ must be visited twice (once going away from the root and once going towards the root).
- For each $s \geq 0$, we then set $C(s)$ to be equal to the distance on the edges of τ from the particle to the root at time s. We take the convention that $C(s) = 0$ for all $s \geq 2|\tau|$.

A *Dyck path* of length $2k$ is a sequence $(x_0, \ldots, x_{2k})$ of non-negative integers so that

(i) $x_0 = 0$, $x_{2k} = 0$ and
(ii) $|x_i - x_{i-1}| = 1$ for all $i = 1, \ldots, 2k$.

We note that if $\tau \in \mathbf{T}_k$, C is its contour function, then $C(0), \ldots, C(2k)$ is a Dyck path of length $2k$.

The following proposition will be proved on Example Sheet 1.

Proposition 2.1. *The map which takes $\tau \in \mathbf{T}_k$ to $(C(0), \ldots, C(2k))$ where C is the contour function of τ is a bijection from $\mathbf{T}_k$ to the set of Dyck paths of length $2k$.*

It follows from Proposition 2.1 that picking $\tau \in \mathbf{T}_k$ uniformly at random is equivalent to picking a Dyck path of length $2k$ uniformly at random. Suppose that $C(0), \ldots, C(2k)$ is a Dyck path of length $2k$. Let (ξ_i) be a sequence of i.i.d. random variables with $\mathbf{P}[\xi_1 = 1] = \mathbf{P}[\xi_1 = -1] = \frac{1}{2}$ so that $S(j) = \sum_{i=1}^{k} \xi_i$ is a simple random walk with $S(0) = 0$. Note that

$$\begin{aligned}\mathbf{P}[S(j) = C(j),\ \forall 0 \leq j \leq 2k] &= \mathbf{P}[S(j) - S(j-1) = C(j) - C(j-1),\ \forall 1 \leq j \leq 2k] \\ &= \mathbf{P}[\xi_j = C(j) - C(j-1),\ \forall 1 \leq j \leq 2k] = 2^{-2k}.\end{aligned}$$

In particular, this probability does not depend on the particular choice of Dyck path. This implies that if we let

$$D_k = \{S(j) \text{ for } 0 \leq j \leq 2k \text{ is a Dyck path of length } 2k\}$$

then the law of $S(j)$ for $0 \leq j \leq 2k$ conditioned on D_k is given by the uniform measure on Dyck paths of length $2k$. This conditional law is also that of a *simple random walk excursion* of length $2k$ since:

(i) $S(0) = S(2k) = 0$
(ii) $S(j) \geq 0$ for all $0 \leq j \leq 2k$.

Thus to describe the continuum limit of a uniformly random element of $\mathbf{T}_k$, we need to describe the continuum limit of a simple random walk excursion, which will be the focus of the next section.

3. The Brownian excursion

Roughly speaking, a Brownian excursion $\mathbf{e}$ is the stochastic process $[0,1] \to \mathbf{R}_+$ which arises by starting with a Brownian motion B_t, $B_0 = 0$, and then conditioning it so that $B_1 = 0$ and $B_t \geq 0$ for all $t \in [0,1]$. This is a zero probability event for Brownian motion, so one must be careful with its definition. One possible way of making this rigorous is as follows. For each $\epsilon > 0$, we can let $A_\epsilon = \{B_1 \in [0,\epsilon],\ B_t \geq -\epsilon\ \forall t \in [0,1]\}$. Then $\mathbf{P}[A_\epsilon] > 0$ so that the conditional law of $B|_{[0,1]}$ given A_ϵ makes sense. It is then possible to show that the conditional law of $B|_{[0,1]}$ given A_ϵ converges as $\epsilon \to 0$. It takes a bit of work to prove that this is the case. We will instead give a different construction of the Brownian excursion which is more direct and will be analogous to the usual construction of Brownian motion.

For $t, x > 0$, we let

$$q_t(x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right).$$

Note that for $x > 0$ fixed, $q_t(x)$ gives the density at t of the first time that a Brownian motion hits x. We also let

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right).$$

Then $p_t(x,y)$ is the usual transition density for Brownian motion. For each $k \in \mathbf{N}$ and $0 < t_1 < \cdots < t_k < 1$, we define the probability measure with density given by

$$\mathbf{BE}_{t_1,\ldots,t_k}(x_1,\ldots,x_k) = 2\sqrt{2\pi} q_{t_1}(x_1) p^*_{t_2-t_1}(x_1,x_2) \cdots p^*_{t_k-t_{k-1}}(x_{k-1},x_k) q_{1-t_k}(x_k)$$

where $p^*_t(x,y) = p_t(x,y) - p_t(x,-y)$ for $t, x, y > 0$. Then $\mathbf{BE}_{t_1,\ldots,t_k}$ defines a consistent family of probability measures. This means that for all $0 < t_1 < \cdots < t_{k+1} < 1$, we have that

$$\int_0^\infty \mathbf{BE}_{t_1,\ldots,t_{k+1}}(x_1,\ldots,x_{k+1}) dx_j = \mathbf{BE}_{t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{k+1}}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{k+1}).$$

Then it is possible to show that there exists a continuous process $\mathbf{e}$ whose finite dimensional distributions are given by $\mathbf{BE}_{t_1,\ldots,t_k}$ (Example Sheet 1) and this is the Brownian excursion.

Remark 3.1. *(1) There are many different constructions of the Brownian excursion. Its most common use in probability theory is in so-called excursion theory, which in the context of*

Brownian motion gives a representation of a Brownian motion as a Poisson point process of Brownian excursions.

(2) The properties of the Brownian excursion are very similar to the properties of Brownian motion. For example, for every $\epsilon > 0$ it is $(\frac{1}{2} - \epsilon)$-Hölder continuous (Example Sheet 1).

(3) The function p^* in the definition of the Brownian excursion has an interpretation. Namely, it is the transition density for $B_{t \wedge \tau}$ where B_t is a Brownian motion with $B_0 > 0$ and $\tau = \inf\{t \geq 0 : B_t = 0\}$ (Example Sheet 1).

4. Real trees and the Gromov-Hausdorff distance

4.1. Real trees.

Definition 4.1. *A compact metric space $(\mathcal{T}, d)$ is called an $\mathbf{R}$-tree if for all $a, b \in \mathcal{T}$ we have that:*

(i) There is a unique isometric map $f_{a,b} \colon [0, d(a,b)] \to \mathcal{T}$ so that $f_{a,b}(0) = a$, $f_{a,b}(d(a,b)) = b$.
(ii) If $q \colon [0,1] \to \mathcal{T}$ is a continuous injective map with $q(0) = a$, $q(1) = b$, then $q([0,1]) = f_{a,b}([0, d(a,b)])$.

A rooted $\mathbf{R}$-tree is an $\mathbf{R}$-tree $(\mathcal{T}, d)$ with a distinguished point $\rho \in \mathcal{T}$ called the root.

Properties (i) and (ii) are the continuum analogs of the connectivity and no-cycles condition in the graph definition of a tree.

Suppose that $(\mathcal{T}, d)$ is a rooted $\mathbf{R}$-tree. We will write $[[a,b]]$ for the range of $f_{a,b}$. If $a, b \in \mathcal{T}$, then there is a unique $c \in \mathcal{T}$ so that $[[\rho, a]] \cap [[\rho, b]] = [[\rho, c]]$. We will use the notation $c = a \wedge b$ and call c the *most recent common ancestor* of a, b. The *multiplicity* of $a \in \mathcal{T}$ is equal to the number of connected components of $\mathcal{T} \setminus \{a\}$. We call $a \in \mathcal{T}$ a *leaf* if it has multiplicity 1.

4.2. **Encoding an R-tree with a continuous function.** Suppose that $g \colon [0,1] \to [0, \infty)$ is a continuous function with $g(0) = g(1) = 0$. For $s, t \in [0,1]$, we let

$$m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r)$$

and

$$d_g(s,t) = g(s) + g(t) - 2 m_g(s,t).$$

Then it follows that $d_g(s,t) = d_g(t,s)$ and $d_g(s,t) \leq d_g(s,u) + d_g(u,t)$ for all $u \in [0,1]$. However, it is not the case in general that $d_g(s,t) = 0$ implies $s = t$ so $d_g(s,t)$ only defines a pseudometric on $[0,1]$. We say that $s \sim t$ if and only if $d_g(s,t) = 0$ which is equivalent to $g(s) = g(t) = m_g(s,t)$. We then set $\mathcal{T}_g = [0,1]/\sim$ and let $\pi_g \colon [0,1] \to \mathcal{T}_g$ be the associated projection map.

Theorem 4.2. *The metric space $(\mathcal{T}_g, d_g)$ is an $\mathbf{R}$ tree.*

This will be proved on Example Sheet 1.

4.3. **The continuum random tree.**

Definition 4.3. *The* continuum random tree *(CRT) is the random* $\mathbf{R}$*-tree which is encoded by the Brownian excursion* $\mathbf{e}$.

One can think of the CRT as the "uniform measure" on $\mathbf{R}$-trees. We will make this rigorous by showing that it arises as the limit of uniformly random discrete trees.

Properties of the CRT correspond to properties of the Brownian excursion $\mathbf{e}$. The following properties will be established on Example Sheet 1.

Theorem 4.4. *Suppose that* $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ *is a CRT and let* $\pi_{\mathbf{e}} \colon [0,1] \to \mathcal{T}_{\mathbf{e}}$ *is the associated projection map. The following properties hold almost surely.*

(i) For Lebesgue a.e. $t \in [0,1]$, $\pi_{\mathbf{e}}(t)$ *is a leaf of* $\mathcal{T}_{\mathbf{e}}$.
(ii) The degree of every $x \in \mathcal{T}_{\mathbf{e}}$ *is at most* 3.
(iii) The set of $x \in \mathcal{T}_{\mathbf{e}}$ *with degree* 3 *is countable.*

4.4. **The Gromov-Hausdorff distance.** The *Gromov-Hausdorff distance* is a metric on the space of compact metric spaces. The starting point for its definition is the *Hausdorff distance* on compact subsets of a metric space. Suppose that (X,d) is a metric space. For $Y \subseteq X$ and $\epsilon > 0$, we let

$$Y_\epsilon = \{x \in X : d(x,Y) \leq \epsilon\}$$

be the ϵ-neighborhood of Y. For $K, K' \subseteq X$ compact, we then define the *Hausdorff distance* between K and K' to be

$$d_{\mathrm{H}}(K,K') = \inf\{\epsilon > 0 : K \subseteq K'_\epsilon,\ K' \subseteq K_\epsilon\}.$$

Theorem 4.5. *The Hausdorff distance* d_{H} *defines a metric on the set of compact subsets of* (X,d).

Proof. Suppose that $K, K', K'' \subseteq X$ are compact. It is obvious from the definition of d_{H} that $d_{\mathrm{H}}(K,K') = d_{\mathrm{H}}(K',K)$. It also not difficult to see from the triangle inequality for d that $d_{\mathrm{H}}(K,K') \leq d_{\mathrm{H}}(K,K'') + d_{\mathrm{H}}(K'',K)$. It is left to show that $d_{\mathrm{H}}(K,K') = 0$ implies that $K = K'$. If $d_{\mathrm{H}}(K,K') = 0$, then we have that $K \subseteq K'_\epsilon$ for all $\epsilon > 0$. Since K' is compact, we have that $K' = \cap_{\epsilon > 0} K'_\epsilon$. Therefore $K \subseteq K'$ and the same argument implies that $K' \subseteq K$ so that $K = K'$. □

If (X,d) is a compact metric space, then it is also possible to show that d_{H} defines a compact metric on the compact subsets of X and the finite sets are dense.

Suppose that (X,d), (X',d') are compact metric spaces. We define the *Gromov-Hausdorff distance* between X and X' as

$$d_{\mathrm{GH}}(X,X') = \inf\{D_{\mathrm{H}}(\varphi(X), \varphi'(X'))\}$$

where the infimum is over all possible metric spaces (E,D) and isometric embeddings $\varphi \colon X \to E$, $\varphi' \colon X' \to E'$. In order to show that $d_{\mathrm{GH}}(X,X')$ is finite, we need to show that there even exists a metric space (E, D) into which both X, X' both embed isometrically. One convenient way to do this is to use that every compact metric space has an embedding into ℓ_∞ (the space of bounded real sequences equipped with the metric $d_\infty((x_n), (y_n)) = \sup_n |x_n - y_n|$). This will be proved on Example Sheet 1.

We say that compact metric spaces (X, d), (X', d') are equivalent if there exists a bijective isometry $\varphi\colon X \to X'$. We let $\mathbf{K}$ be the set of all equivalence classes of compact metric spaces.

Theorem 4.6. *We have that d_{GH} defines a metric on $\mathbf{K}$.*

Remark 4.7. *It is also possible to show that $(\mathbf{K}, d_{\mathrm{GH}})$ is complete and separable and that the finite metric spaces are dense.*

Proof of Theorem 4.6. Suppose that (X, d), (X', d') are compact metric spaces. It is obvious that $d_{\mathrm{GH}}(X, X') = d_{\mathrm{GH}}(X', X)$. That d_{GH} satisfies the triangle inequality will be proved on Example Sheet 1. It is left to show that $d_{\mathrm{GH}}(X, X') = 0$ implies that X is equivalent to X'. Suppose that $d_{\mathrm{GH}}(X, X') = 0$. Let (x_n) be a countable dense subset of X. Fix $\epsilon > 0$. Then there exist a metric space (E, D) and isometries $\varphi\colon X \to E$, $\varphi'\colon X' \to E$ so that $D_{\mathrm{H}}(\varphi(X), \varphi'(X')) \leq \epsilon$. Therefore for every $n \in \mathbf{N}$, there exists $y_n' \in \varphi'(X')$ so that $D(\varphi(x_n), y_n') \leq \epsilon$. Let $\psi_\epsilon(x_n) = (\varphi')^{-1}(y_n')$. Since (X', d') is compact, it follows that for each fixed $n \in \mathbf{N}$ we can find a sequence (ϵ_j) so that $\epsilon_j > 0$ and $\epsilon_j \to 0$ as $j \to \infty$ so that $\psi_{\epsilon_j}(x_n)$ converges as $j \to \infty$. By passing to a diagonal subsequence, we can in fact assume that $\psi_{\epsilon_j}(x_n)$ converges for every $n \in \mathbf{N}$. Define

$$\psi(x_n) = \lim_j \psi_{\epsilon_j}(x_n).$$

Note from the construction that we have that

$$|d'(\psi_{\epsilon_j}(x_n), \psi_{\epsilon_j}(x_m)) - d(x_n, x_m)| \leq 2\epsilon_j \quad \text{for all} \quad n, m \in \mathbf{N}.$$

Taking a limit as $j \to \infty$, we see that

$$d'(\psi(x_n), \psi(x_m)) = d(x_n, x_m) \quad \text{for all} \quad n, m \in \mathbf{N}.$$

That is, ψ is an isometry on (x_n). We can extend ψ to an isometry of X into X' as follows. Suppose that $x \in X$ and (x_{n_j}) is a subsequence of (x_n) which converges to x. Then it follows that $(\psi(x_{n_j}))$ is a Cauchy sequence and we set $\psi(x)$ to be its limit. Note that the limit does not depend on the choice of subsequence (x_{n_j}) because if (x_{m_k}) is another subsequence of (x_n) which converges to x then the subsequence obtained by interleaving (x_{n_j}) and (x_{m_k}) also converges to x and ψ applied to it will be Cauchy hence have the same limit as that of $(\psi(x_{n_j}))$. If we perform an analogous construction with the roles of X and X' swapped, we can also construct an isometry ψ' of X' into X so that $\psi' \circ \psi$ is the identity so that X and X' are equivalent. ☐

We call a metric space (X, d) pointed if it has a distinguished point $\rho \in X$. Two pointed metric spaces (X, d, ρ) and (X', d', ρ') are said to be equivalent if there exists a bijective isometry $\varphi\colon X \to X'$ with

$\varphi(\rho) = \rho'$. Let $\mathbf{K}^\bullet$ denote the set of equivalence classes of pointed compact metric spaces. Then one can generalize the Gromov-Hausdorff metric to $\mathbf{K}^\bullet$ by setting

$$d_{\mathrm{GH}}((X,\rho),(X',\rho')) = \inf\{D_{\mathrm{H}}(\varphi(X),\varphi'(X')) + D(\varphi(\rho),\varphi'(\rho'))\}$$

where the infimum is over all metric spaces (E,D) and isometries $\varphi\colon X \to E$, $\varphi'\colon X' \to E$.

There is an equivalent formulation of the Gromov-Hausdorff metric which does not require one to consider embeddings into a larger ambient space. The starting point is the definition of a *correspondence* between metric spaces X, X', which is a subset $\mathcal{R}$ fo $X \times X'$ so that every $x \in X$ there exists $x' \in X'$ so that $(x,x') \in \mathcal{R}$ and vice-versa. The *distortion* of a correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d(x,y) - d'(x',y')| : (x,x'),(y,y') \in \mathcal{R}\}.$$

The following will be proved on Example Sheet 1.

Theorem 4.8. *Suppose that $X, X' \in \mathbf{K}$. Then*

$$d_{\mathrm{GH}}(X,X') = \frac{1}{2}\inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R})$$

where the infimum is over all correspondences of X and X'. If X, X' are pointed by ρ, ρ', respectively, then the same is true except the infimum is over all correspondences X and X' which contain (ρ,ρ').

Corollary 4.9. *Suppose that $f, g\colon [0,1] \to [0,\infty)$ are continuous functions with $f(0) = f(1) = g(0) = g(1) = 0$. Then*

$$d_{\mathrm{GH}}(\mathcal{T}_f, \mathcal{T}_g) \leq 2\|f - g\|_\infty.$$

Proof. Let $\pi_f\colon [0,1] \to \mathcal{T}_f$, $\pi_g\colon [0,1] \to \mathcal{T}_g$ be the canonical projection maps. Let

$$\mathcal{R} = \{(a,a') : \exists t \in [0,1] \quad \text{such that} \quad a = \pi_f(t),\ a' = \pi_g(t)\}.$$

Note that $(\pi_f(0),\pi_g(0)) \in \mathcal{R}$ (i.e., the roots of $\mathcal{T}_f$ and $\mathcal{T}_g$). Then $\mathcal{R}$ is a correspondence of $\mathcal{T}_f$ and $\mathcal{T}_g$. Suppose that $(a,a'),(b,b') \in \mathcal{R}$. Then there exists $s,t \in [0,1]$ so that $\pi_f(s) = a$, $\pi_g(s) = a'$, $\pi_f(t) = b$, $\pi_g(t) = b'$. Recall that $d_f(a,b) = f(s) + f(t) - 2m_f(s,t)$ and $d_g(a',b') = g(s) + g(t) - 2m_g(s,t)$. Therefore $|d_f(a,b) - d_g(a',b')| \leq 4\|f-g\|_\infty$ which implies that $\mathrm{dis}(\mathcal{R}) \leq 4\|f-g\|_\infty$. □

5. Convergence of discrete trees to the continuum random tree

Theorem 5.1. *For each $k \in \mathbf{N}$, let τ_k be uniformly distributed over $\mathbf{T}_k$ and let C_k be its contour function. Then $((2k)^{-1/2}C(2kt))_{0\leq t\leq 1}$ converges to the Brownian excursion as $k \to \infty$ in the sense of weak convergence of probability measures on $C([0,1],[0,\infty))$.*

Combining Theorem 5.1 with Corollary 4.9 implies that τ_k equipped with its graph metric rescaled by $(2k)^{-1/2}$ converges to the continuum random tree as $k \to \infty$ weakly in the Gromov-Hausdorff topology.

Before we proceed to the proof of Theorem 5.1, let us first describe the setup. Let $S(n)$ be a simple random walk on $\mathbf{Z}$ with $S(0) = 0$. Let $\sigma = \min\{n \geq 1 : S(n) = -1\}$. Then we need to show that $((2k)^{-1/2} S(\lfloor 2kt \rfloor))_{0 \leq t \leq 1}$ under the law $\mathbf{P}[\cdot \,|\, \sigma = 2k+1]$ converges to the Brownian excursion. There are two main steps:

(1) Convergence of the finite dimensional distributions and
(2) Tightness (Example Sheet 1)

We will being by proving the convergence of a single marginal by showing that for each $t \in (0,1)$ we have that

$$\lim_{k \to \infty} \sqrt{2k} \mathbf{P}[S(\lfloor 2kt \rfloor) = \lfloor x\sqrt{2k} \rfloor \text{ or } \lfloor x\sqrt{2k} \rfloor \,|\, \sigma = 2k+1] = 4\sqrt{2\pi} q_t(x) q_{1-t}(x)$$

uniformly for any x in a compact subset of $(0, \infty)$.

We will need two lemmas as input. The first is a special case of the local central theorem, which will be proved on Example Sheet 1.

Lemma 5.2. *For every $\epsilon > 0$,*

$$\lim_{n \to \infty} \sup_{x \in \mathbf{R}} \sup_{s \geq \epsilon} \left| \sqrt{n} \mathbf{P}[S(\lfloor ns \rfloor) = \lfloor x\sqrt{n} \rfloor \quad \text{or} \quad \lfloor x\sqrt{n} \rfloor + 1] - 2p_s(0,x) \right| = 0.$$

Lemma 5.3. *For all $\ell, n \in \mathbf{N}$ we have that*

$$\mathbf{P}_\ell[\sigma = n] = \frac{\ell+1}{n} \mathbf{P}_\ell[S(n) = -1].$$

Proof. We have that

$$\mathbf{P}_\ell[\sigma = n] = \frac{1}{2} \mathbf{P}_\ell[S(n-1) = 0, \ \sigma > n-1].$$

We also have that

$$\begin{aligned}
\mathbf{P}_\ell[S(n-1) = 0, \ \sigma > n-1] &= \mathbf{P}_\ell[S(n-1) = 0] - \mathbf{P}_\ell[S(n-1) = 0, \ \sigma \leq n-1] \\
&= \mathbf{P}_\ell[S(n-1) = 0] - \mathbf{P}_\ell[S(n-1) = -2, \ \sigma \leq n-1] \quad \text{(by reflection)} \\
&= \mathbf{P}_\ell[S(n-1) = 0] - \mathbf{P}_\ell[S(n-1) = -2].
\end{aligned}$$

The last equality follows since $S(n-1) = -2$ implies $\sigma \leq n-1$. We thus have that

$$\mathbf{P}_\ell[\sigma = n] = \frac{1}{2} \left(\mathbf{P}_\ell[S(n-1) = 0] - \mathbf{P}_\ell[S(n-1) = -2] \right).$$

Since $S(n-1)$, $S(n)$ are binomial random variables, it follows that the right hand side above is equal to

$$\frac{\ell+1}{n} \mathbf{P}_\ell[S(n-1) = -1].$$

□

Proof of Theorem 5.1. For $1 \leq i \leq 2k$ and $\ell \geq 1$, we have that

$$\mathbf{P}[S(i) = \ell \,|\, \sigma = 2k+1] = \frac{\mathbf{P}[S(i) = \ell, \ \sigma = 2k+1]}{\mathbf{P}[\sigma = 2k+1]}.$$

Applying the Markov property at time i, the numerator is equal to

$$\mathbf{P}[S(i) = \ell,\ \sigma > i]\mathbf{P}_\ell[\sigma = 2k + 1 - i].$$

We also have that

$$\mathbf{P}[S(i) = \ell,\ \sigma > i] = 2\mathbf{P}_\ell[\sigma = i + 1].$$

Therefore

$$\begin{aligned}\mathbf{P}[S(i) = \ell \,|\, \sigma = 2k + 1] &= \frac{2\mathbf{P}_\ell[\sigma = i + 1]\mathbf{P}_\ell[\sigma = 2k + 1 - i]}{\mathbf{P}[\sigma = 2k + 1]} \\ &= \frac{2(2k + 1)(\ell + 1)^2}{(i + 1)(2k + 1 - i)} \frac{\mathbf{P}_\ell[S(i + 1) = -1]\mathbf{P}_\ell[S(2k + 1 - i) = -1]}{\mathbf{P}[S(2k + 1) = -1]} \quad \text{(by Lemma 5.3)}.\end{aligned}$$

The proof for the convergence of the first order marginals then follows by applying the local CLT (Lemma 5.2). □

6. Planar maps

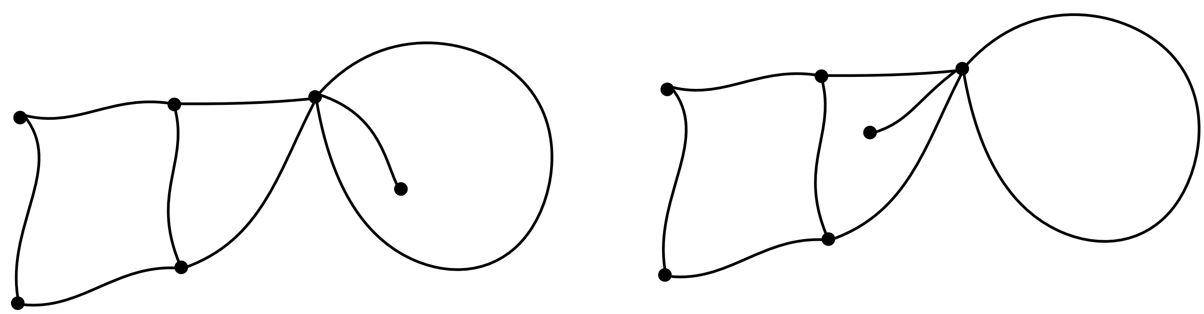

Figure 6.1. Two planar maps m and m' which are associated with isomorphic graphs but are not equivalent as maps.

6.1. Basic definitions.

A *planar map* is a graph drawn on the two-dimensional sphere $\mathbf{S}^2$. As *oriented edge* is a continuous map $e\colon [0, 1] \to \mathbf{S}^2$ such that either e is injective or $e|_{[0,1)}$ is injective and $e(0) = e(1)$ (corresponding to a loop). We will always consider oriented edges modulo reparameterization. The *origin* (resp. *target*) of an oriented edge e is $e^- = e(0)$ (resp. $e^+ = e(1)$). The time-reversal $\overline{e}$ of e is defined by $\overline{e}(\cdot) = e(1 - \cdot)$. An *edge* is a pair $\mathbf{e} = \{e, \overline{e}\}$ where e is an oriented edge. We will sometimes abuse notation and not distinguish between edges and oriented edges for quantities which do not depend on the direction of time. The *interior* of e is defined to be $e(0, 1)$. An *embedded graph* on $\mathbf{S}^2$ is a graph $G = (V, E)$ (assumed to be finite but multiple edges and self-loops allowed) such that:

- V is a finite subset of $\mathbf{S}^2$
- E is a finite set of edges
- The vertices incident to $e \in E$ are $e^+, e^- \in V$
- The interior of an edge $e \in E$ does not intersect V or the other edges in E

The *support* of G is

$$\mathrm{supp}(G) = V \cup \bigcup_{e \in E} e([0,1]).$$

The *faces* of G are the components of $\mathbf{S}^2 \setminus \mathrm{supp}(G)$. Let $m = (V, E)$ be a (planar) map. We let $\overrightarrow{E} = \{e \in \mathbf{e} : \mathbf{e} \in E\}$ be the set of oriented edges. For each oriented edge $e \in \overrightarrow{E}$, there is a face f_e of m which is to the left of e. The *degree* of F is defined be

$$\deg(f) = \#\{e \in \overrightarrow{E} : f_e = f\}.$$

For a vertex $u \in V$, we also set

$$\deg(u) = \#\{e \in \overrightarrow{E} : e^- = u\}.$$

A *rooted map* is a pair (m, e) where $m = (V, E)$ is a map and $e \in \overrightarrow{E}$ is the root (oriented edge). We say that maps m, m' are equivalent if there exists an orientation preserving homeomorphism $\phi \colon \mathbf{S}^2 \to \mathbf{S}^2$ which makes m to m'. We emphasize that even if the graph structure associated with two maps m, m' are isomorphic, it may not be that the maps are equivalent. We also say that rooted maps (m, e), (m', e') are equivalent if there exists an orientation preserving homeomorphism $\phi \colon \mathbf{S}^2 \to \mathbf{S}^2$ which takes m to m' and e to e'. Throughout, we will consider equivalent maps to be the same.

We let $\mathcal{M}_n$ be the set of rooted maps with n edges. It was proved by Tutte that

$$\#\mathcal{M}_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}.$$

We let $\mathcal{Q}_n$ be the set of rooted quadrangulations (i.e., maps whose faces all have degree 4) with n faces. It turns out that $\mathcal{M}_n$ and $\mathcal{Q}_n$ are in bijective correspondence through the so-called "trivial bijection". Given $m \in \mathcal{M}_n$, one can produce an element $q \in \mathcal{Q}_n$ using the following steps:

- Add to each face f of m a new vertex $v(f)$
- For each face f of m, connect $v(f)$ to the vertices incident to f
- Let q be the rooted quadrangulation whose vertices consist of the vertices of m, the new vertices $v(f)$, and the new edges (but not the original edges of m). Make the root edge of q be the first edge with the same origin as the root of m in the clockwise direction.

6.2. **Cori-Vauquelin-Schaeffer bijection.** Suppose that $\tau \in \mathbf{T}_k$. A *labelling* of τ is a map $\ell \colon \tau \to \mathbf{Z}$ such that:

- $\ell(\emptyset) = 0$
- For all $v \in \tau \setminus \{\emptyset\}$, we have that $\ell(v) - \ell(\pi(v)) \in \{-1, 0, 1\}$

We let $\mathbf{LT}_k$ be the set of labelled trees with k edges. Note that

$$\#\mathbf{LT}_k = 3^k \#\mathbf{T}_k = \frac{3^k}{k+1} \binom{2k}{k}.$$

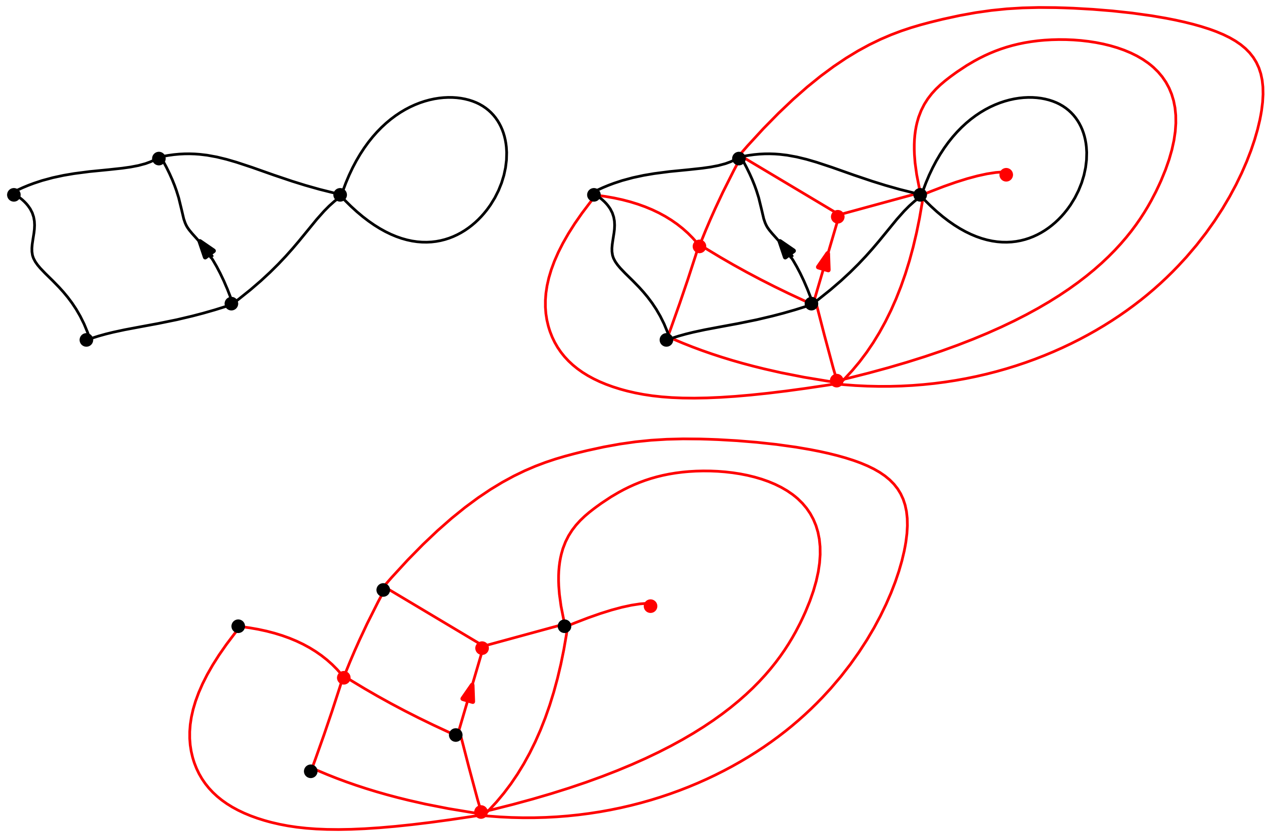

Figure 6.2. An illustration of the "trivial bijection" between $\mathcal{M}_n$ and $\mathcal{Q}_n$. **Left:** An element m of $\mathcal{M}_7$ with vertices and edges drawn in black. **Right:** The corresponding element q of $\mathcal{Q}_7$ superimposed on m with edges in red and vertices in black and red. **Bottom:** The quadrangulation q on its own. We note that m can be recovered from q by coloring the origin (resp. target) of the root edge black (resp. red) and the rest of the vertices accordingly (as q is bipartite), adding a black edge between the opposing black vertices in each face, and then erasing the red edges.

The reason for this is that there are 3 possible choices for the change in label across each edge.

We will now describe the *Cori-Vauquelin-Schaeffer* (CVS) bijection, which serves to construct a quadrangulation with k faces from a labelled plane tree with k edges. Suppose that $(\ell, \tau) \in \mathbf{LT}_k$. We view τ as a planar map (i.e., embedded into $\mathbf{S}^2$). Let $\phi = v_0, v_1, \ldots, v_{2n}$ be the contour exploration of τ. For each i, we let e_i be the edge from v_i to v_{i+1}. We write

$$\ell(e_i) = \ell(e_i^-) = \ell(v_i).$$

We also let

$$s(i) = \min\{j > i : \ell(e_j) = \ell(e_i) - 1\}.$$

We call $s(i)$ the "successor" of i. We will sometimes abuse notation and write $s(e_i)$ for $e_{s(i)}$. We now construct a new planar map as follows. We add a point $v_* \in \mathbf{S}^2$ in $\mathbf{S}^2 \setminus \mathrm{supp}(\tau)$. For each i, we also define the "corner" of e_i to be a simply connected neighborhood of e^- in $\mathbf{S}^2 \mathrm{supp}(\tau)$ which

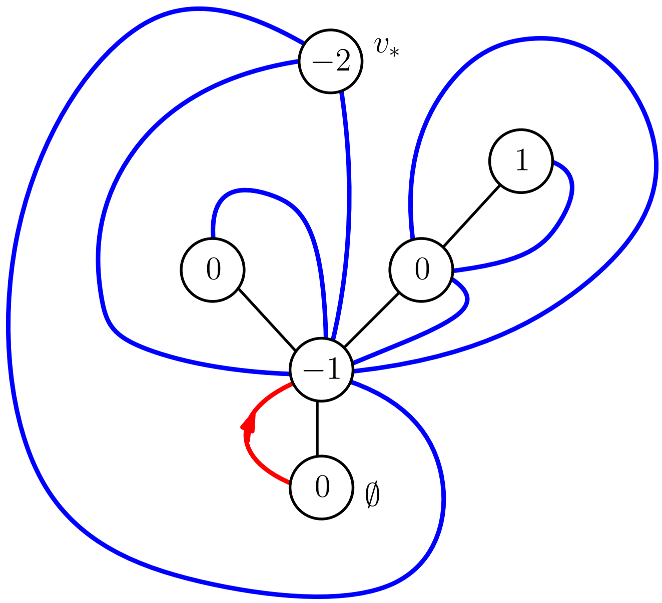

FIGURE 6.3. A labeled tree with 4 edges together with the element of $\mathcal{Q}_4^\bullet$ constructed using the CVS bijection. The edges of the quadrangulation are shown in blue and red, where the red edge is the root, all other edges are blue, and we have taken $\epsilon = 1$. The edges of the labelled tree are shown in black.

is to the left of e_i^-. We will also sometimes abuse notation and write e_i for the corner associated with e_i. For each $i \in \{0, \ldots, 2n-1\}$, we then draw an edge from the corner e_i to the corner $s(e_i)$ in $\mathbf{S}^2 \setminus (\{v_*\} \cup \mathrm{supp}(\tau))$. If $\ell(e_i) = \min\{j : \ell(e_j)\}$, then we instead drawn an edge from the corner e_i to v_*. Suppose that we have $\epsilon \in \{-1, 1\}$. We then root the map by taking the distinguished edge to be:

- From e_0 to $s(e_0)$ if $\epsilon = 1$
- From $s(e_0)$ to e_0 if $\epsilon = -1$.

Then we obtain a rooted, pointed map (m, e, v_*). We let $\mathcal{Q}_n^\bullet$ be the set of rooted, pointed quadrangulations with n faces.

Theorem 6.1. *The procedure Q_{CVS} described above defines a bijection $\mathbf{LT}_n \times \{-1, 1\} \to \mathcal{Q}_n^\bullet$.*

Corollary 6.2. *We have that*

$$\#\mathcal{Q}_n^\bullet = \frac{2 \cdot 3^n}{n+1} \binom{2n}{n}.$$

We also have that

$$\#\mathcal{M}_n = \#\mathcal{Q}_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}.$$

Proof. The first part of the corollary is immediate from Theorem 6.1. The second part of the corollary follows from the trivial bijection and the first part since Euler's formula (Example Sheet 1) implies that if $q \in \mathcal{Q}_n$ then q has $n+2$ vertices. In particular, $\#\mathcal{Q}_n^\bullet = (n+2)\#\mathcal{Q}_n$. $\square$

To prove Theorem 6.1, we will first show that $Q_{\mathrm{CVS}}(\ell,\tau,\epsilon)$ is in $\mathcal{Q}_n^\bullet$. We will describe the inverse procedure.

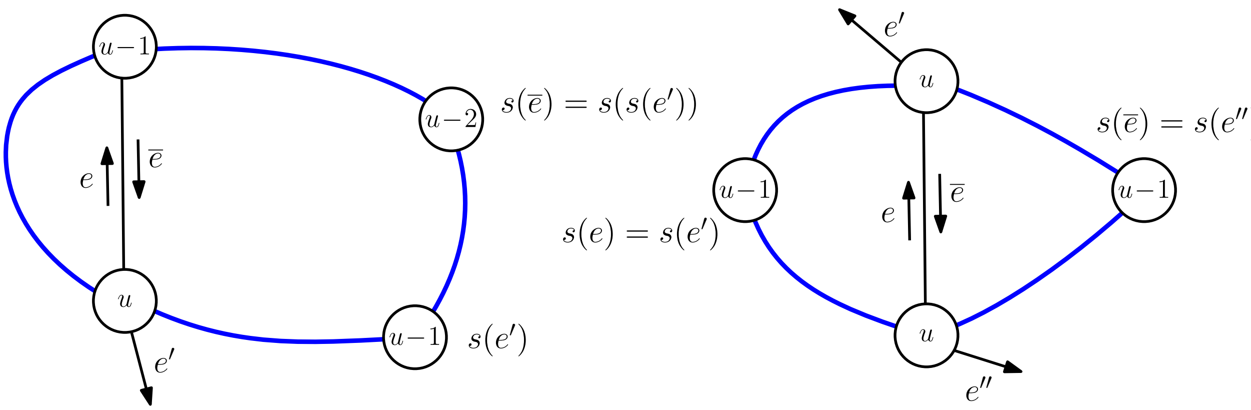

FIGURE 6.4. **Left:** Illustration of the proof that one obtains a quadrilateral in the first case for the CVS procedure. Here, $s(e')$ comes before $s(\overline{e})$ in contour order and therefore $s(s(e')) = s(\overline{e})$. This is an example of a "simple face". Note that the edge of τ in the face is incident to the vertex on the face boundary with largest label. **Right:** Illustration of the second case. This is an example of a "confluent face". Notice that the edge of τ connects the vertices on the face boundary with the largest label.

Lemma 6.3. *Let* $(\ell,\tau,\epsilon) \in \mathbf{LT}_n \in \{-1,1\}$ *and let* $q = Q_{\mathrm{CVS}}(\ell,\tau,\epsilon)$ *constructed using the CVS procedure. Then* $q \in \mathcal{Q}_n^\bullet$.

Proof. We begin by observing that q is connected. This follows since every vertex in q has a path of vertices connecting it to v_* which is obtained by applying the successor operation.

Consider an edge of τ which corresponds to the oriented edges $e, \overline{e}$. We assume that $\ell(e^+) = \ell(e^-) - 1$. Then $s(e)$ is incident to e^+ and the CVS procedure gives an arc starting from e^- and ending at e^+. Let e' be the corner following $\overline{e}$ in the contour exploration around τ. Then $\ell(e') = \ell(e^-) = \ell(\overline{e}) + 1$ and $s(\overline{e}) = s(s(e'))$. Indeed, $s(e')$ is the first corner coming after e' in contour order and with label $\ell(e') - 1 = \ell(e) - 1$ while $s(s(e'))$ is the first corner after e' with label $\ell(e) - 2$. Therefore it has to be the first corner coming after $\overline{e}$ with label $\ell(e) - 2 = \ell(\overline{e}) - 1$. Therefore the arcs joining the corners e to $s(e)$, $\overline{e}$ to $s(\overline{e})$, e' to $s(e')$, and $s(e')$ to $s(s(e'))$ form a quadrilateral which contains $\{e, \overline{e}\}$ and no other edge of τ.

If $\ell(e^+) = \ell(e^-) + 1$, then exactly the same argument as above applies.

If $\ell(e^+) = \ell(e^-)$, and we let e', e'', respectively, be the corners following $e, \overline{e}$, respectively, in the contour exploration of τ, then $\ell(e) = \ell(e') = \ell(\overline{e}) = \ell(e'')$. Therefore $s(e) = s(e')$ and $s(\overline{e}) = s(e'')$.

This gives that $\{e, \bar{e}\}$ is the diagonal across a quadrilateral with arcs connecting e to $s(e)$, e' to $s(e') = s(e)$, $\bar{e}$ to $s(\bar{e})$, and e'' to $s(e'') = s(\bar{e})$. We note that q has $2n$ edges (one per corner) and $n+2$ vertices. Euler's formula (Example Sheet 1) thus implies that q has n faces. $\square$

We will now describe the inverse to Q_{CVS}. Suppose that $q \in \mathcal{Q}_n^\bullet$. We let $\phi(x) = d(x, v_*)$ where d denotes the graph distance, $x \in \mathbf{V}(q)$, and v_* is the marked point in q. Then we have the following observations:

- q is a bipartite graph (Example Sheet 1)
- If $x, y \in \mathbf{V}(q)$ are joined by an edge, then $|\phi(x) - \phi(y)| = 1$. If we color v_* black, then $\phi(x)$ is even if and only if $x \in \mathbf{V}(q)$ is black.
- Around each face, we have vertices x_1 (black), y_1 (white), x_2 (black), y_2 (white). Then at least one of $\phi(x_1) = \phi(x_2)$ or $\phi(y_1) = \phi(y_2)$ is satisfied.

We call a face $f \in \mathbf{F}(q)$ *simple* if precisely one of the above equalities holds and we call f *confluent* if both equalities hold.

We label $\mathbf{V}(q)$ with the function $\phi(x) - \phi(e^-)$ where e is the root edge of q. We then define a map T_{CVS} as follows.

- In each confluent face, we draw a diagonal connecting the opposing vertices which have maximal label.
- In each simple face f, we take the edge with maximal label which has f on its right.

Lemma 6.4. *Suppose that $q \in \mathcal{Q}_n^\bullet$ and $(\ell, \tau) = T_{\text{CVS}}(q)$ as defined above. Then $(\ell, \tau) \in \mathbf{LT}_n$.*

Proof. Suppose that $x \in \mathbf{V}(q)$ with $x \neq v_*$. Then it has a neighbor y with $\ell(y) = \ell(x) - 1$. Then $\{x, y\}$ can be incident to:

- A confluent face
- A simple face where x is the maximal label (among vertices of the face)
- Two simple faces where x has an intermediate label (among the vertices of the faces)

(See Figure 6.5.) In each of the three possibilities, x is incident to an edge of τ. Therefore $\mathbf{V}(\tau) = \mathbf{V}(q) \setminus \{v_*\}$. This implies that $\#\mathbf{V}(\tau) = \#\mathbf{V}(q) - 1 = n+1$. We also have that $\#\mathbf{E}(\tau) = n$ since we defined on edge of τ for each face of q. Moreover, since q is planar we have that τ is planar.

We will now prove that τ is a tree. Suppose that τ has a cycle C and let u be the value of the smallest label along C. Either all of the labels on C are equal to u or the cycle contains edges with labels $(u, u+1)$ and $(u+1, u)$. (See Figure 6.6.) In either case, it follows from the procedure for constructing τ that there would be $x, y \in \mathbf{V}(q)$, one in each of the two complementary components of C, with label $u-1$. Note that the shortest path from x to v_* or y to v_* has to pass through the cycle. This cannot happen because the distances along the shortest such path to v_* decrease by 1 at each step.

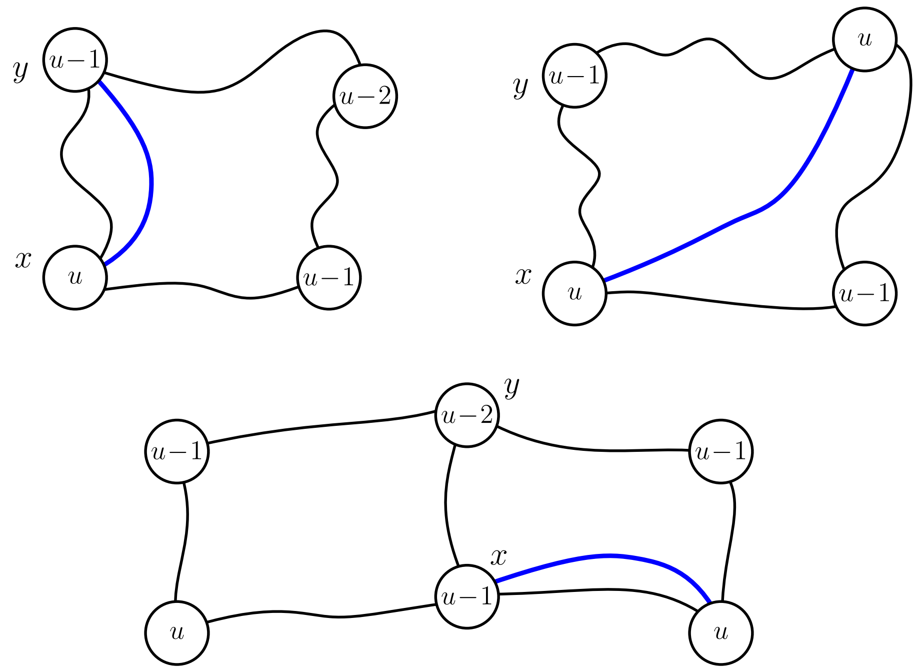

Figure 6.5. Illustration of the three cases from the proof of Lemma 6.4.

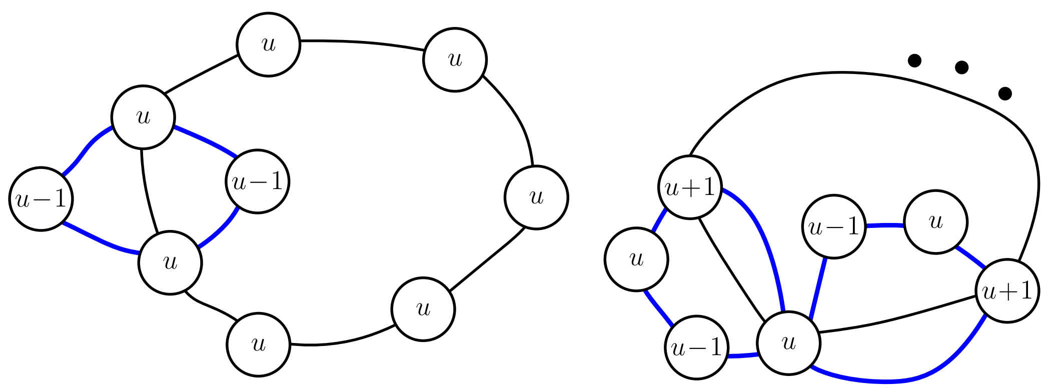

Figure 6.6. Illustration of the proof that τ cannot have a cycle in the proof of Lemma 6.4. In each of the two cases, both of the complementary components of C will contain vertices x, y with label $u-1$.

We have shown that τ does not have cycles. This implies that it is a forest (a finite union of trees). Since it has n edges and $n+1$ vertices, it is in fact a tree. □

If one examines the proof of Lemma 6.3, then one can notice that $T_{\mathrm{CVS}}(Q_{\mathrm{CVS}}(\ell,\tau,\epsilon)) = (\ell,\tau,\epsilon)$ for all $(\ell,\tau,\epsilon) \in \mathbf{LT}_n \in \{-1,1\}$. It was proved by Tutte that $\#\mathbf{LT}_n \times \{-1,1\} = \#\mathcal{Q}_n^\bullet$, which implies that Q_{CVS} is surjective. However, one can show directly that $Q_{\mathrm{CVS}}(T_{\mathrm{CVS}}(q)) = q$ for all $q \in \mathcal{Q}_n^\bullet$ (Example Sheet 2).

7. Random planar maps

7.1. **General comments.** Recall that one can think of Brownian motion as being in some sense the "uniform measure" on continuous paths. The way that this is made rigorous sense of is by discretizing and considering simple random walk on $\mathbf{Z}$ which is the uniform measure on paths in $\mathbf{Z}$ which change by 1 in each time step. Since simple random walk converges to Brownian motion in the scaling limit, one can think of Brownian motion as the uniform measure on continuous paths because it is the scaling limit of the uniform measure on paths in $\mathbf{Z}$. Similarly, the continuum random tree (CRT) can be thought of as the "uniform measure" on planar trees. We made sense of this earlier by discretizing and showing that it arises as the scaling limit of uniformly random (discrete) plane trees (i.e., elements of $\mathbf{T}_k$). Finally, the Brownian map can be thought of as the "uniform measure" on surfaces homeomorphic to $\mathbf{S}^2$. The way that this made sense of is by discretizing the problem and considering random quadrangulations and showing that they converge in the limit to the Brownian map. Note that one can think of a quadrangulation as corresponding to a surface by identifying each of its faces with a copy of $[0,1]^2$ and then gluing together adjacent copies of $[0,1]^2$ according to Euclidean length.

$$
\begin{tikzcd}
\text{Random plane tree} \arrow[rr, leftrightarrow] \arrow[d, "\substack{\text{Scaling limit}\\ \text{Gromov-Hausdorff}\\ \text{topology}}"'] & & \text{Contour function} \arrow[d, "\substack{\text{Scaling limit}\\ \|\cdot\|_\infty \text{ topology}}"] \\
\text{Continuum random tree} & & \text{Brownian excursion} \arrow[ll]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Random quadrangulation} \arrow[rr, leftrightarrow] \arrow[d, "\substack{\text{Scaling limit}\\ \text{Gromov-Hausdorff}\\ \text{topology}}"'] & & \text{Labelled tree} \arrow[d, "\substack{\text{Scaling limit}\\ \|\cdot\|_\infty \text{ topology}}"] \\
\text{Brownian map} & & \text{Brownian snake} \arrow[ll]
\end{tikzcd}
$$

FIGURE 7.1. **Top:** Schematic illustration of the strategy to prove that uniformly random plane trees converge in the scaling limit to the Continuum random tree in the Gromov-Hausdorff topology. **Bottom:** Schematic illustration of the strategy to prove that uniformly random quadrangulations converge in the scaling limit to the Brownian map in the Gromov-Hausdorff topology. Unlike the case of trees, the Gromov-Hausdorff convergence for quadrangulations does not immediately follow from the scaling limit result for the encoding process (labelled trees to the Brownian snake). This in fact a very difficult result which was proved relatively recently in works of Le Gall and Miermont (2011).

Why does one study random planar maps and random surfaces? This has become a very active topic of research in probability theory in the last 20 or so years. There are a number of different motivations for this, but let us focus on one here. There are many different models from statistical mechanics that probabilists have been studying in the last century (Brownian motion, the percolation model, the Ising model, the self-avoiding walk, etc...) In two dimensions, physicists developed an accurate picture as to how they should behave in the 1970s-2000s using non-rigorous methods. A number of these predictions were later verified by mathematicians using Schramm-Loewner evolution (SLE), the next topic in this course.

One very famous example of this type is Mandelbrot's conjecture, as described in Example 1.2. Recall that this states that the dimension of the outer boundary of a planar Brownian motion is $4/3$ (roughly speaking, this means that the number of disks of radius ϵ necessary to cover it grows to leading order like $\epsilon^{-4/3}$ as $\epsilon \to 0$). Proving Mandelbrot's conjecture as well as the derivation of a number of other properties of planar Brownian motion amounts to studying "non-intersection probabilities" for Brownian motion. This means the following. Suppose that we have k independent planar Brownian motions $X^1, \ldots, X^k$ which start from equally spaced points on $\partial B(0,\epsilon)$. For each j, let $\tau_j = \inf\{t \geq 0 : X^j(t) \notin B(0,1)\}$. The question is how unlikely is it that the ranges $X^j([0,\tau_j])$ are pairwise disjoint for $1 \leq j \leq k$. Since planar Brownian motion is neighborhood recurrent, this is a are event and its probability turns out to behave like a power of ϵ as $\epsilon \to 0$.

One can also formulate the problem in terms of k independent simple random walks in $\mathbf{Z}^2$ starting from points which are of constant order distance from the origin and ask for the probability that they travel distance n without intersecting. When phrased in this way, the problem is purely combinatorial and amounts to counting numbers of non-intersecting paths in $\mathbf{Z}^2$. This turns out to be a very difficult question. Physicists realized that this type of counting question becomes a lot easier when the underlying graph is random (i.e., simple random walk on a random planar map). When one solves the problem in the setting of a random planar map, one gets a different answer than if one were to solve it on $\mathbf{Z}^2$ since the underlying graph is different. They also developed a method (the so-called KPZ relation) for converting the probabilities of this type computed in the setting of random graphs to the setting of $\mathbf{Z}^2$ (or another planar lattice). Since the methods employed are non-rigorous, this only leads to a prediction. Many of these predictions were then verified rigorously by mathematicians (including Mandelbrot's conjecture) using SLE.

7.2. **The Brownian snake.** Recall that a random quadrangulation can be encoded in terms of a random labelled tree (an element of $\mathbf{LT}_n$) using the CVS bijection. One can think of sampling a random element of $\mathbf{LT}_n$ in two steps:

(1) Sample $\tau \in \mathbf{T}_n$ uniformly at random
(2) Given τ, sample the labels ℓ uniformly at random

As one travels along a branch of τ, the labels change by -1, 0, or 1 along each edge. This means that the labels along a given branch evolve as a random walk. If one considers multiple branches, then the random walks for each branch are the same until the branches separate.

The Brownian snake is a way to construct the continuous version of this process. Roughly, the way works is the following:

(1) Sample a CRT
(2) Given the realization of the CRT, sample Brownian motions on the branches which are coupled together to be the same until the branch points after which they become independent.

We are now going to describe how to make this construction rigorous. Suppose that $g\colon [0,1] \to \mathbf{R}_+$ is a continuous function with $g(0) = g(1) = 0$. We will further assume that g is Hölder continuous, which we recall means that there exist constants $C > 0$ and $\alpha \in (0,1]$ so that $|g(s) - g(t)| \leq C|t-s|^\alpha$ for all $s,t \in [0,1]$. We also let $m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r)$. The following lemma will be proved on Example Sheet 2.

Lemma 7.1. *The function $m_g(s,t)$ is non-negative definite. That is, for all $s_1, \ldots, s_n \in [0,1]$ and $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$, we have that*

$$\sum_{i,j=1}^n \lambda_i \lambda_j m_g(s_i, s_j) \geq 0.$$

Lemma 7.1 implies that there exists a mean-zero Gaussian process Z on $[0,1]$ with $\mathbf{E} Z_s Z_t = m_g(s,t)$. The process Z is the "Brownian snake driven by g". We note that

$$\begin{aligned} \mathbf{E}|Z_s - Z_t|^2 &= \mathbf{E} Z_s^2 + \mathbf{E} Z_t^2 - 2\mathbf{E} Z_s Z_t \\ &= g(s) + g(t) - 2m_g(s,t) \\ &\leq 2C|s-t|^\alpha. \end{aligned}$$

This further implies that for every $n \in \mathbf{N}$ we have that

$$\mathbf{E}|Z_s - Z_t|^{2n} \leq c_n |s-t|^n$$

for a constant $c_n > 0$. Therefore the Kolmogorov-Centsov continuity criterion implies that Z has a modification which is $(\alpha/2 - \epsilon)$-Hölder continuous for every $\epsilon > 0$.

The "Brownian snake" is the process Z obtained by:

(1) Sampling a Brownian excursion $\mathbf{e}$
(2) Given $\mathbf{e}$, taking Z to be the Brownian snake driven by $\mathbf{e}$.

Sine the Brownian excursion is $(1/2 - \epsilon)$-Hölder continuous for every $\epsilon > 0$, it follows that the Brownian snake is $(1/4 - \epsilon)$-Hölder continuous for every $\epsilon > 0$.

7.3. **Convergence of labeled trees to the Brownian snake.** Suppose that $(\ell, \tau) \in \mathbf{LT}_n$. Let $v_0, \ldots, v_{2n}$ be its contour exploration. For each $j \in \{0, \ldots, 2n\}$, we let $V(j) = \ell(v_j)$. We then extend V to be a function on $[0, 2n]$ by linear interpolation. We call V the *contour label function* for (ℓ, τ).

Theorem 7.2. *For each $k \in \mathbf{N}$, let (ℓ_k, τ_k) be uniformly distributed on $\mathbf{LT}_k$. Let C_k, V_k be the contour and label contour functions for (ℓ_k, τ_k). Then we have that*

$$\left(\frac{1}{\sqrt{2k}} C_k(2kt), \left(\frac{9}{8k}\right)^{1/4} V_k(2kt)\right)_{0 \leq t \leq 1} \xrightarrow{d} (\mathbf{e}, Z)$$

where $(\mathbf{e}, Z)$ is the Brownian snake and the convergence is in the sense of distributions on $C([0,1], \mathbf{R}^2)$.

Proof. We have already showed that $((2k)^{-1/2} C_k(2kt))_{0 \leq t \leq 1} \xrightarrow{d} \mathbf{e}$. By the Skorokhod representation theorem for weak convergence, we can put the sequence (τ_k) and $\mathbf{e}$ on a common probability space so that $((2k)^{-1/2} C_k(2kt))_{0 \leq t \leq 1} \to \mathbf{e}$ a.s. with respect to the $\| \cdot \|_\infty$ distance.

The remainder of the proof has two steps:

(1) Show that the finite dimensional distributions converge
(2) Establish tightness (Example Sheet 2)

To prove the convergence of the finite dimensional distributions, we need to show that for each $0 < t_1 < \cdots < t_r < 1$ we have that

$$\left(\frac{1}{\sqrt{2k}} C_k(2kt_i), \left(\frac{9}{8k}\right)^{1/4} V_k(2kt_i)\right)_{1 \leq i \leq r} \xrightarrow{d} (\mathbf{e}_{t_i}, Z_{t_i})_{1 \leq i \leq r}. \tag{7.1}$$

We note that we have both

$$|C_k(2kt_i) - C_k(\lfloor 2kt_i \rfloor)| \leq 1 \quad \text{and} \quad |V_k(2kt_i) - V_k(\lfloor 2kt_i \rfloor)| \leq 1.$$

Therefore it suffices to prove (7.1) with the integer part $\lfloor 2kt_i \rfloor$ in place of $2kt_i$.

We will give the proof in the case that $r = 1$. Fix $0 < t < 1$ and let $T_k = \lfloor 2kt \rfloor$. Recall that the labels ℓ_k of τ_k are given by performing a random walk on the branches of τ_k with increments in $\{-1, 0, 1\}$. Therefore we can write

$$(C_k(T_k), V_k(T_k)) \stackrel{d}{=} \left(C_k(T_k), \sum_{i=1}^{C_k(T_k)} \eta_i\right)$$

where the (η_i) are i.i.d. random variables which are uniform on $\{-1, 0, 1\}$ and are independent of the τ_k. By the central limit theorem, we have that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i \xrightarrow{d} \left(\frac{2}{3}\right)^{1/2} N \quad \text{where} \quad N \sim N(0,1).$$

This implies that

$$\Phi(n, \lambda) = \mathbf{E}\left[\exp\left(\frac{i\lambda}{\sqrt{n}} \sum_{j=1}^{n} \eta_j\right)\right] \to \exp\left(-\frac{\lambda^2}{3}\right) \quad \text{as} \quad n \to \infty.$$

Therefore if we condition on τ_k we have for all $\lambda, \theta \in \mathbf{R}$ that

$$\mathbf{E}\left[\exp\left(\frac{i\lambda}{\sqrt{2k}}C_k(T_k) + \frac{i\theta}{\sqrt{C_k(T_k)}}\sum_{j=1}^{C_k(T_k)}\eta_j\right)\right] = \mathbf{E}\left[\exp\left(\frac{i\lambda}{\sqrt{2k}}C_k(T_k)\right)\Phi(C_k(T_k),\theta)\right].$$

Since we have that $C_k(T_k)/\sqrt{2k} \to \mathbf{e}(t)$ a.s. (recall the usage of the Skorokhod representation theorem at the beginning of the proof), it thus follows (using also dominated convergence) that the above converges as $k \to \infty$ to

$$\mathbf{E}[\exp(i\lambda\mathbf{e}(t))]\exp(-\theta^2/3).$$

Combining, we have show that

$$\left(\frac{1}{\sqrt{2k}}C_k(T_k), \frac{1}{\sqrt{C_k(T_k)}}\sum_{j=1}^{C_k(T_k)}\eta_j\right) \xrightarrow{d} (\mathbf{e}(t), (2/3)^{1/2}N)$$

where $N \sim N(0,1)$ is independent of $\mathbf{e}$.

We now write

$$\left(\frac{1}{\sqrt{2k}}C_k(T_k), \left(\frac{9}{8k}\right)^{1/4}V_k(T_k)\right) \stackrel{d}{=} \left(\frac{C_k(T_k)}{\sqrt{2k}}, \left(\frac{3}{2}\right)^{1/2}\left(\frac{C_k(T_k)}{\sqrt{2k}}\right)^{1/2}\frac{1}{\sqrt{C_k(T_k)}}\sum_{j=1}^{C_k(T_k)}\eta_j\right)$$
$$\to (\mathbf{e}(t), \sqrt{\mathbf{e}(t)}N) \stackrel{d}{=} (\mathbf{e}(t), Z_t).$$

This completes the proof of the convergence of the first order marginal. General finite dimensional distributions are proved in the same manner, but there is more bookkeeping. We will therefore omit the details. ☐

Recall that if $(\ell, \tau, \epsilon) \in \mathbf{LT}_n \times \{-1, 1\}$ and $q \in \mathcal{Q}_n^\bullet$ is the corresponding quadrangulation, then for $x \in \mathbf{V}(q) = \mathbf{V}(\tau) \cup \{v_*\}$ we have that $\ell(x) - \ell(v_*)$ gives the distance of x to v_* in q. therefore the convergence statement from the theorem implies that the typical diameter of $q \in \mathcal{Q}_n^\bullet$ chosen uniformly at random is of order $n^{1/4}$. In fact,

$$\left(\frac{9}{8n}\right)^{1/4}\max_{x\in\mathbf{V}(q)}\big(\ell(x) - \ell(v_*)\big) \xrightarrow{d} \sup_t Z_t - \inf_t Z_t.$$

Let us make a few further comments:

(1) It is possible to deduce from the theorem the tightness of $(q_n, (9/(8n))^{1/4}d_n)$, where q_n is uniformly random in $\mathcal{Q}_n^\bullet$ and d_n is its graph metric, with respect to the Gromov-Hausdorff topology. It is then a difficult result of Le Gall and Miermont that the subsequential limit exists as a true limit (giving rise to the Brownian map).
(2) There is a generalization of the CVS bijection to $2p$-angulations called the BDG bijection and also to triangulations. An argument analogous to the theorem above also gives the convergence of the analog of the contour label function to the Brownian snake.

(3) In general, one has the "same behavior" for planar maps chosen uniformly at random from any reasonable class (in the same way that many different types of random walks converge in the limit to Brownian motion).
(4) There are also many other types of random planar maps that one studies which are *not* chosen uniformly at random. In these cases, the behavior is very different. There are a number of bijections that one uses to study these types of maps and they are always based on an encoding in terms of trees.

8. Conformal mapping review

Suppose that U, V are domains in $\mathbf{C}$ and that $f\colon U \to V$ is a map. We say that f is *holomorphic* if it is complex differentiable, i.e., for each $z \in U$ then limit

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \quad \text{exists.}$$

A *conformal transformation* is a map which is a bijection (also sometimes called a "conformal equivalence" or just "conformal").

A domain $U \subseteq \mathbf{C}$ is called *simply connected* if $\mathbf{C} \setminus U$ is connected. Important examples of simply connected domains include the complex plane $\mathbf{C}$, the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, and the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.

Theorem 8.1 (Riemann mapping theorem)**.** *Suppose that U is a simply connected domain with $U \neq \mathbf{C}$ and $z \in U$. Then there exists a unique conformal transformation $f\colon \mathbf{D} \to U$ with $f(0) = z$ and $f'(0) > 0$.*

We will not give a proof of the Riemann mapping theorem here. It can be found in most complex analysis textbooks. An immediate consequence of the Riemann mapping theorem is that any two simply connected domains which are both distinct from $\mathbf{C}$ can be mapped to each other using a conformal transformation.

Corollary 8.2. *If U, V are simply connected domains with $U, V \neq \mathbf{C}$ and $z \in U$ and $w \in V$, then there exists a unique conformal transformation $f\colon U \to V$ with $f(z) = w$ and $f'(z) > 0$.*

8.1. **Examples.**

Conformal transformations of $\mathbf{D}$. Suppose that $U = \mathbf{D}$ and $z \in \mathbf{D}$. Then $f\colon \mathbf{D} \to \mathbf{D}$ given by

$$f(w) = \frac{w + z}{1 + \overline{z}w}$$

is the unique conformal transformation with $f(0) = z$ and $f'(0) > 0$. More generally, every conformal transformation $f\colon \mathbf{D} \to \mathbf{D}$ is of the form

$$f(w) = \lambda \frac{w - z}{\overline{z}w - 1}$$

where $\lambda \in \partial\mathbf{D}$ and $z \in \mathbf{D}$. So, there is a three-real-parameter family of such maps (z corresponds to two parameters and λ to one).

The map $f \colon \mathbf{H} \to \mathbf{D}$ given by

$$f(z) = \frac{z-i}{z+i}$$

is a conformal transformation. It is the so-called Cayley transform. Its inverse $g \colon \mathbf{D} \to \mathbf{H}$ is given by

$$g(w) = \frac{i(1+w)}{1-w}$$

and is also a conformal transformation.

The conformal transformations $\mathbf{H} \to \mathbf{H}$ consist of the maps of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbf{R}$ with $ad - bc = 1$.

More generally, if U, V are simply connected domains with $U, V \neq \mathbf{C}$, then there is a three-parameter family of conformal transformations $f \colon U \to V$.

Here is another important example which motivates the definition of SLE. For each $t \geq 0$, let $H_t = \mathbf{H} \setminus [0, 2\sqrt{t}i]$. Let $g_t \colon H_t \to \mathbf{H}$ be the map $z \mapsto \sqrt{z^2 + 4t}$. Then g_t is a conformal transformation $H_t \to \mathbf{H}$.

We make two observations about the family of conformal maps (g_t). First, we have that

$$|g_t(z) - z| = |\sqrt{z^2+4t} - z| \to 0 \quad \text{as} \quad z \to \infty.$$

That is, "g_t looks like the identity map at ∞."

Second, we have that

$$\partial_t g_t(z) = \frac{1}{2\sqrt{z^2+4t}} \cdot 4 = \frac{2}{g_t(z)}.$$

So, for each $z \in \mathbf{H}$ fixed we have that $g_t(z)$ solves the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z)}, \quad \text{with} \quad g_0(z) = z. \tag{8.1}$$

For each $z \in \mathbf{H}$, the basic existence and uniqueness theorem for ODEs implies that (8.1) has a unique solution up until the denominator on the right hand side explodes, i.e.

$$\tau(z) = \inf\{t \geq 0 : \mathrm{Im}(g_t(z)) = 0\}.$$

In other words, the family of conformal transformations (g_t) are characterized by (8.1). In particular, the curve $\gamma(t) = 2\sqrt{t}i$ is encoded by (8.1). This is a special case of Loewner's theorem.

Here is a preview for later on in the course. Suppose that γ is any simple curve (i.e., non-self-intersecting) in $\mathbf{H}$ starting from 0. For each $t \geq 0$, let g_t be the unique conformal transformation which maps $H_t := \mathbf{H} \setminus \gamma([0,t])$ to $\mathbf{H}$ with $|g_t(z) - z| \to \infty$. (We will later prove that there indeed does exist a unique such conformal transformation.) Then Loewner's theorem states that there exists a continuous, real-valued function W such that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad \text{with} \quad g_0(z) = z.$$

This is the so-called *chordal Loewner equation.* Using this equation, we see that there is a correspondence between simple curves in $\mathbf{H}$ and continuous, real-valued functions.

The case $\gamma(t) = 2\sqrt{t}i$ corresponds to $W = 0$.

SLE_κ corresponds to the case $W = \sqrt{\kappa}B$ where B is a standard Brownian motion.

9. Brownian motion, harmonic functions, and conformal maps

Recall that $f = u + iv$ is holomorphic if and only if u satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{9.1}$$

One important consequence of the Cauchy-Riemann equations is that if f is holomorphic then u, v are harmonic. This means that

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \quad \text{and} \quad \Delta v = 0.$$

Indeed,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}.$$

We will now recall a few results which were proved in Advanced Probability which serve to relate harmonic functions and Brownian motion. Throughout, we say that a process $B = B^1 + iB^2$ is a *complex Brownian motion* if B^1, B^2 are independent standard Brownian motions in $\mathbf{R}$.

Theorem 9.1. *Let u be a harmonic function on a bounded domain D which is continuous on $\overline{D}$. Fix $z \in D$ and let $\mathbf{P}_z$ be the law of a complex Brownian motion B starting from z and let $\tau = \inf\{t \geq 0 : B_t \notin D\}$. Then*

$$u(z) = \mathbf{E}_z[u(B_\tau)].$$

Proof. This was proved in Advanced Probability. Another proof based on Itô's formula will be given in Stochastic Calculus. □

Theorem 9.2 (Mean-value property for harmonic functions)**.** *In the setting of the prevoius theorem if $z \in D$ and $r > 0$ are such that $B(z, r) = \{w \in \mathbf{C} : |w - z| < r\} \subseteq D$, then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta.$$

Proof. This was proved in Advanced Probability. □

Theorem 9.3 (Maximum principle)**.** *Suppose that u is harmonic in a domain D. If u attains its maximum at an interior point in D, then u is constant.*

Proof. Assume that u attains its maximum at $z_0 \in D$. Let $D_0 = \{z \in D : u(z) = u(z_0)\}$. Then $D_0 \neq \emptyset$ since $z_0 \in D_0$. The continuity of u in D implies that D_0 is (relatively) closed in D. Suppose that $z \in D_0$ and $r > 0$ is such that $B(z,r) \subseteq D$. Then $u|_{\partial B(z,r)} = u(z_0)$ for otherwise there exists $w \in \partial B(z,r)$ and $\epsilon > 0$ such that u is at most $u(z_0) - \epsilon$ on $B(w,\epsilon)$ which, by the mean-value property, would contradict that $u(z) = u(z_0)$. Combining this with Theorem 9.1 implies that u is constant on $B(z,r)$. Therefore D_0 is open hence $D_0 = D$. □

Theorem 9.4 (Maximum modulus principle)**.** *Let D be a domain and let $f \colon D \to \mathbf{C}$ be a holomorphic map. If $|f|$ attains its maximum in the interior of D, then f is constant.*

Proof. Assume that f attains its maximum at $z_0 \in D$. Let K be compact in D with $z_0 \in K$. Assume further that the interior of K is connected and that K is the closure of its interior. By replacing f with $f + M$ for $M \in \mathbf{R}$ sufficiently large, we can assume that $|f| \neq 0$ on K. Note that $\log|f|$ is a harmonic function on K. As $|f|$ attains its maximum in D on K, it follows that $\log|f|$ does as well, hence $\log|f|$ is constant on K by the maximum principle. Therefore $|f|$ is constant on K as well. Since K was an arbitrary compact subset of D containing z_0 (which is connected and is the closure of its interior), we deduce that $|f|$ is constant on all of D. This implies that $f(D)$ is contained in a circle in $\mathbf{C}$ hence the Lebesgue measure of $f(D)$ is equal to 0. It is easy to see that if $f'(z) \neq 0$ for some $z \in D$, then the area of $f(D)$ is strictly positive. Therefore $f'(z) = 0$ for all $z \in D$, which implies that f is constant on D. □

Theorem 9.5 (Schwarz Lemma)**.** *Suppose that $f \colon \mathbf{D} \to \mathbf{D}$ is a holomorphic map with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbf{D}$. If $|f(z)| = |z|$ for some $z \in \mathbf{D}$, then there exists $\theta \in \mathbf{R}$ so that $f(w) = we^{i\theta}$ (i.e., f is a rotation map).*

Proof. Let

$$g(z) = \begin{cases} f(z)/z & \text{if} \quad z \neq 0, \\ f'(0) & \text{if} \quad z = 0. \end{cases}$$

Then g is a holomorphic map on $\mathbf{D}$ and $|g(z)| \leq 1$ for all $z \in \mathbf{D}$ by the maximum modulus principle. If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbf{D} \setminus \{0\}$ then the maximum modulus principle implies that there exists $c \in \mathbf{C}$ such that $g(z) = c$ for all $z \in \mathbf{D}$. As $|g(z_0)| = 1$ it follows that $|c| = 1$. That is, there exists $\theta \in \mathbf{R}$ so that $c = e^{i\theta}$. Hence, $f(w) = e^{i\theta} w$ as claimed. □

10. Half-plane capacity

Definition 10.1. A set $A \subseteq \mathbf{H}$ is called a *compact* $\mathbf{H}$*-hull* if $A = \mathbf{H} \cap \overline{A}$, $\overline{A}$ is compact, and $\mathbf{H} \setminus A$ is simply connected. We let $\mathcal{Q}$ be the collection of compact $\mathbf{H}$-hulls.

In this section, we will be interested in

- Analyzing the "correct" conformal transformation $g_A\colon \mathbf{H} \setminus A \to \mathbf{H}$ and
- A notion of "size" for $A \in \mathcal{Q}$ (half-plane capacity).

Proposition 10.2. *For each $A \in \mathcal{Q}$, there exists a unique conformal transformation $g_A\colon \mathbf{H}\setminus A \to \mathbf{H}$ with $|g_A(z) - z| \to 0$ as $z \to \infty$.*

In order to prove Proposition 10.2, we will need to make use of the so-called Schwarz reflection principle.

Proposition 10.3 (Schwarz reflection principle)**.** *Let $D \subseteq \mathbf{H}$ be a simply connected domain and let $\phi\colon D \to \mathbf{H}$ be a conformal transformation which is bounded on bounded sets. Then ϕ extends by reflection to a conformal transformation on $D^* = D \cup \{\overline{z} : z \in D\} \cup \{x \in \partial\mathbf{H} : D \text{ is a neighborhood of } x \text{ in } \mathbf{H}\}$ by setting $\phi(\overline{z}) = \overline{\phi(z)}$.*

We will not provide a proof of Proposition 10.3.

Proof of Proposition 10.2. The Riemann mapping theorem implies that there exists a conformal transformation $g\colon \mathbf{H} \setminus A \to \mathbf{H}$. By post-composing $\mathbf{H}$ with a conformal transformation $\mathbf{H} \to \mathbf{H}$ if necessary, we may assume without loss of generality that $|g(z)| \to \infty$ as $|z| \to \infty$ (i.e., g fixes ∞). By Schwarz reflection, we can extend g to a conformal transformation defined on $\mathbf{C} \setminus (\{\overline{z} : z \in A\} \cup \overline{A})$ by setting $g(\overline{z}) = \overline{g(z)}$. By performing a series expansion for $1/g(1/z)$, we see that g admits the Laurent expansion

$$g(z) = b_{-1}z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

If $z \in \mathbf{R}$, then $\overline{z} = z$ and $g(z) = g(\overline{z}) = \overline{g(z)}$. That is, if $z \in \mathbf{R} \setminus \overline{A}$ then $g(z) \in \mathbf{R}$. Consequently,

$$b_{-1}z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} = \overline{b_{-1}}z + \overline{b_0} + \sum_{n=1}^{\infty} \frac{\overline{b_n}}{z^n} \quad \text{for all} \quad z \in \mathbf{R} \setminus \overline{A}.$$

This implies that $b_j = \overline{b_j}$ for each j. In other words, each b_j is real. Set

$$g_A(z) = \frac{g(z) - b_0}{b_{-1}}.$$

As $b_{-1}, b_0 \in \mathbf{R}$, we have that $g_A\colon \mathbf{H} \setminus A \to \mathbf{H}$ is a conformal transformation with $|g_A(z) - z| \to 0$ as $z \to \infty$. This completes the proof of existence.

To see the uniqueness, suppose that $\widetilde{g}_A\colon \mathbf{H} \setminus A \to \mathbf{H}$ is another conformal transformation such that $|\widetilde{g}_A(z) - z| \to 0$ as $z \to \infty$. Then $\widetilde{g}_A \circ g_A^{-1}$ is a conformal transformation $\mathbf{H} \to \mathbf{H}$. This implies that there exists $a, b, c, d \in \mathbf{R}$ with $ad - bc = 1$ such that

$$\widetilde{g}_A \circ g_A^{-1}(z) = \frac{az + b}{cz + d}.$$

Since $|\widetilde{g}_A \circ g_A^{-1}(z) - z| \to 0$ as $z \to \infty$, it follows that $a = c = 1$ and $b = d = 0$. That is, $\widetilde{g}_A \circ g_A^{-1}(z) = z$ which implies that $\widetilde{g}_A = g_A$. □

Definition 10.4. Suppose that $A \in \mathcal{Q}$. The *half-plane capacity* of A is defined by

$$\mathrm{hcap}(A) = \lim_{z\to\infty} z(g_A(z) - z).$$

Equivalently, we have that

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + \sum_{n=2}^{\infty} \frac{b_n}{z^n}.$$

One should think of hcap(A) as a notion of "size" for A. We will shortly show that it is non-negative and monotone.

Example 10.5. Recall that $z \mapsto \sqrt{z^2+4t}$ is a conformal transformation $\mathbf{H} \setminus [0, 2\sqrt{t}i] \to \mathbf{H}$ with $|\sqrt{z^2+4t} - z| \to 0$ as $z \to \infty$. Note that

$$\sqrt{z^2+4t} = z + \frac{2t}{z} + \cdots .$$

Therefore $\mathrm{hcap}([0, 2\sqrt{t}i]) = 2t$.

Example 10.6. The map $z \mapsto z + 1/z$ maps $\mathbf{H} \setminus \overline{\mathbf{D}} \to \mathbf{H}$ and $|(z + 1/z) - z| \to 0$ as $z \to \infty$. Therefore $\mathrm{hcap}(\mathbf{H} \cap \overline{\mathbf{D}}) = 1$.

We are now going to collect several properties of the half-plane capacity.

(i) **Scaling.** Suppose that $r > 0$, $A \in \mathcal{Q}$. Then $\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A)$ and $g_{rA}(z) = rg_A(z/r)$.
(ii) **Translation invariance.** Suppose that $x \in \mathbf{R}$ and $A \in \mathcal{Q}$. Then $\mathrm{hcap}(A + x) = \mathrm{hcap}(A)$ and $g_{A+x}(z) = g_A(z - x) + x$.
(iii) **Monotonicity.** Suppose that $A, \widetilde{A} \in \mathcal{Q}$ with $A \subseteq \widetilde{A}$. Then

$$\mathrm{hcap}(\widetilde{A}) = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(\widetilde{A} \setminus A)).$$

Upon showing that hcap ≥ 0, this will imply that $\mathrm{hcap}(\widetilde{A}) \geq \mathrm{hcap}(A)$. That is, hcap is monotone.

By combining the scaling and monotonicity properties of the half-plane capacity, we note that if $A \in \mathcal{Q}$ and $A \subseteq r\overline{\mathbf{D}} \cap \mathbf{H}$, then we have that

$$\mathrm{hcap}(A) \leq \mathrm{hcap}(r\overline{\mathbf{D}} \cap \mathbf{H}) = r^2\mathrm{hcap}(\overline{\mathbf{D}} \cap \mathbf{H}) = r^2.$$

We now turn to derive a representation for the half-plane capacity in terms of Brownian motion, which in particular implies that the half-plane capacity is non-negative.

Proposition 10.7. *Suppose that $A \in \mathcal{Q}$, B is a complex Brownian motion, and $\tau = \inf\{t \geq 0 : B_t \notin \mathbf{H} \setminus A\}$ is the first exit time of B from $\mathbf{H} \setminus A$.*

(i) *For all* $z \in \mathbf{H} \setminus A$*,* $\mathrm{Im}(z - g_A(z)) = \mathbf{E}_z[\mathrm{Im}(B_\tau)]$.
(ii) $\mathrm{hcap}(A) = \lim_{y\to\infty} y\mathbf{E}_{iy}[\mathrm{Im}(B_\tau)]$.
(iii) $\mathrm{hcap}(A) = \frac{2}{\pi}\int_0^\pi \mathbf{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]\sin(\theta)d\theta$.

Proof. Note that $\phi(z) = \mathrm{Im}(z - g_A(z))$ is harmonic in $\mathbf{H} \setminus A$ as it is the imaginary part of a complex differentiable function. As $g_A(z) = z + \mathrm{hcap}(A)/z + \cdots$ and $\mathrm{Im}(g_A(z)) = 0$ for $z \in \partial(\mathbf{H} \setminus A)$, it follows that ϕ is bounded and continuous. Therefore (i) follows from Theorem 9.1.

Note that

$$\begin{aligned}\mathrm{hcap}(A) &= \lim_{z\to\infty} z(g_A(z) - z)\\ &= \lim_{y\to\infty} iy(g_A(iy) - iy).\end{aligned}$$

The proof of Proposition 10.2 implies that $\mathrm{hcap}(A)$ is real (as the coefficients in the series expansion of g_A are real). Taking real parts of both sides, we thus see that

$$\mathrm{hcap}(A) = \lim_{y\to\infty} y\mathrm{Im}(iy - g_A(iy)).$$

Therefore (ii) follows from (i).

Part (iii) is on Example Sheet 2. □

Before we proceed to derive some estimates for g_A, we pause to discuss the conformal invariance of Brownian motion. Roughly, this says that if B is a complex Brownian motion and f is a conformal transformation, then the random process $f(B)$ is a Brownian motion up to a random time-change. This statement can be checked directly in the special case that $f(z) = cz + d$ for $c, d \in \mathbf{C}$ (i.e., f can be thought of as first performing a rotation, then a scaling, then a translation) because one can check directly from the definition of complex Brownian motion then it is rotationally invariant, scale invariant (up to a time change), and translation invariant. Conformal transformations locally behave like such f, which is why this fact is intuitive. We now give a formal statement:

Theorem 10.8. *Let* $D, \widetilde{D}$ *be domains and let* $f\colon D \to \widetilde{D}$ *be a conformal transformation. Let* $B, \widetilde{B}$ *be complex Brownian motions starting from* $z \in D$*,* $\widetilde{z} = f(z) \in \widetilde{D}$*, respectively. Let*

$$\tau = \inf\{t \geq 0 : B_t \notin D\} \quad \text{and} \quad \widetilde{\tau} = \inf\{t \geq 0 : \widetilde{B}_t \notin \widetilde{D}\}$$

be the exit times of $B, \widetilde{B}$ *from* $D, \widetilde{D}$*, respectively. Set*

$$\tau' = \int_0^\tau |f'(B_s)|^2 ds \quad \text{and} \quad \sigma(t) = \inf\left\{s \geq 0 : \int_0^s |f'(B_r)|^2 dr = t\right\} \quad \text{for} \quad t < \tau'.$$

With $B'_t = f(B_{\sigma(t)})$*, we have that*

$$(\tau', B'_t : t < \tau') \stackrel{d}{=} (\widetilde{\tau} : \widetilde{B}_t : t < \widetilde{\tau}).$$

Theorem 10.8 will be given as a problem on an example sheet in Stochastic Calculus. It is proved by applying Itô's formula, the Cauchy-Riemann equations, and the Lévy characterization of Brownian motion.

We can use Theorem 10.8 to deduce the form of the exit distribution of a complex Brownian motion from a simply connected domain D. Since we will only be concerned with exit distributions, we emphasize that the random time-change in Theorem 10.8 will not play a role. Here are a few cases that will be important for what follows:

- If B is a complex Brownian motion in $\mathbf{D}$ starting from 0, then its first exit distribution is given by the uniform distribution on $\partial\mathbf{D}$. This follows because complex Brownian motion is rotationally invariant.
- Using Theorem 10.8 and applying a conformal transformation $\mathbf{D} \to \mathbf{D}$ which takes 0 to a given point $z \in \mathbf{D}$, one can show that the density (with respect to Lebesgue measure on $\partial\mathbf{D}$) of the first exit distribution of a complex Brownian motion starting from z at the point $e^{i\theta} \in \partial\mathbf{D}$ is given by
$$\frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2} \quad \text{for} \quad \theta \in [0,2\pi).$$
This is on Example Sheet 2.
- Again using Theorem 10.8, one can see that the first exit distribution of a complex Brownian motion starting from $z = x+iy \in \mathbf{H}$ from $\mathbf{H}$ has density with respect to Lebesgue measure on $\mathbf{R}$ given by
$$\frac{1}{\pi}\frac{y}{(x-u)^2+y^2} \quad \text{for} \quad u \in \partial\mathbf{H}.$$
This is also on Example Sheet 2.

For $A \in \mathcal{Q}$, we let
$$\mathrm{rad}(A) = \sup\{|z| : z \in A\}.$$
That is, $\mathrm{rad}(A)$ is the diameter of the smallest ball centered at the origin which contains A.

Proposition 10.9. *There exists $c > 0$ such that for all $A \in \mathcal{Q}$ and $|z| \geq 2\mathrm{rad}(A)$ we have that*
$$\left| g_A(z) - z - \frac{\mathrm{hcap}(A)}{z} \right| \leq c\frac{\mathrm{rad}(A)\mathrm{hcap}(A)}{|z|^2}.$$

Proof. By scaling, we may assume without loss of generality that $\mathrm{rad}(A) = 1$. Throughout, we let
$$h(z) = z + \frac{\mathrm{hcap}(A)}{z} - g_A(z).$$
Our goal is then to bound $|h(z)|$. We will proceed by bounding the modulus of the imaginary part of h and then deduce the bound for h itself using the Cauchy-Riemann equations. To this end, we let
$$v(z) = \mathrm{Im}(h(z)) = \mathrm{Im}(z - g_A(z)) - \frac{\mathrm{Im}(z)\mathrm{hcap}(A)}{|z|^2}.$$

Let B be a complex Brownian motion and let $\sigma = \inf\{t \geq 0 : B_t \notin \mathbf{H} \setminus \overline{\mathbf{D}}\}$. We also let $\tau = \inf\{t \geq 0 : B_t \notin \mathbf{H} \setminus A\}$. For $\theta \in [0, \pi]$, we let $p(z, e^{i\theta})$ be the density with respect to Lebesgue measure at $e^{i\theta}$ for B_σ. It follows from the strong Markov property for B at time σ together with part (i) of Proposition 10.7 that

$$\mathrm{Im}(z - g_A(z)) = \int_0^\pi \mathbf{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]p(z, e^{i\theta})d\theta.$$

Recall that

$$p(z, e^{i\theta}) = \frac{2}{\pi}\frac{\mathrm{Im}(z)}{|z|^2}\sin(\theta)\left(1 + O(|z|^{-1})\right) \quad \text{(Example Sheet 2, Problem 9)} \tag{10.1}$$

$$\mathrm{hcap}(A) = \frac{2}{\pi}\int_0^\pi \mathbf{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]\sin(\theta)d\theta \quad \text{(part (iii) of Proposition 10.7)}. \tag{10.2}$$

We thus have that

$$\begin{aligned}
|v(z)| &= \left|\mathrm{Im}(z - g_A(z)) - \frac{\mathrm{Im}(z)}{|z|^2}\mathrm{hcap}(A)\right| \\
&= \left|\int_0^\pi \mathbf{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]p(z, e^{i\theta})d\theta - \frac{2}{\pi}\frac{\mathrm{Im}(z)}{|z|^2}\int_0^\pi \mathbf{E}_{e^{i\theta}}\mathrm{Im}(B_\tau)\sin(\theta)d\theta\right| \quad \text{(by (10.2))} \\
&\leq c\frac{\mathrm{hcap}(A)\mathrm{Im}(z)}{|z|^3} \quad \text{(by (10.1))},
\end{aligned}$$

where $c > 0$ is a constant.

As v is harmonic (as it is the imaginary part of a complex differentiable function), it follows from Example Sheet 1, Problem 8 that we have for a constant $c > 0$ both

$$|\partial_x v(z)| \leq c\frac{\mathrm{hcap}(A)}{|z|^3} \quad \text{and} \quad |\partial_y v(z)| \leq c\frac{\mathrm{hcap}(A)}{|z|^3}.$$

By the Cauchy-Riemann equations, this implies that (possibly increasing the value of c)

$$|h'(z)| \leq c\frac{\mathrm{hcap}(A)}{|z|^3}. \tag{10.3}$$

Hence,

$$\begin{aligned}
|h(iy)| &= \left|\int_y^\infty h'(is)ds\right| \quad \text{(as } h(iy) \to 0 \text{ as } y \to \infty) \\
&\leq \int_y^\infty |h'(is)|ds \\
&\leq c\frac{\mathrm{hcap}(A)}{y^2} \quad \text{(by (10.3))},
\end{aligned}$$

with another possible increase in the value of c in the last inequality. This proves the bound for $z = iy$. For general $z = re^{i\theta}$ with $r \geq 2\mathrm{rad}(A)$, we can integrate along $\partial(r\mathbf{D})$ using the bound (10.3) to see that

$$|h(z)| \leq |h(ir)| + c\frac{\mathrm{hcap}(A)}{r^2},$$

which completes the proof. □

11. The chordal Loewner equation

For $T > 0$, we also let $\mathcal{A}_T$ be the collection of families of compact $\mathbf{H}$-hulls which satisfy (??)–(??) but are only defined on the interval $[0, T]$ (so that $\mathcal{A} = \mathcal{A}_\infty$).

Theorem 11.1. *Suppose that (A_t) is in $\mathcal{A}$ with $A_0 = \emptyset$. For each $t \geq 0$, let $g_t = g_{A_t}$. There exists $U\colon [0, \infty) \to \mathbf{R}$ continuous such that*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Proof. Note that $\cap_{s>t}\overline{g_t(A_s)}$ contains a single point since (A_t) is locally growing. Call this point U_t. It is not difficult to see that in fact U_t is continuous in t since (A_t) is locally growing.

Recall from Proposition 10.9 that if $A \in \mathcal{Q}$ then

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O\left(\frac{\mathrm{hcap}(A)\mathrm{rad}(A)}{|z|^2}\right). \tag{11.1}$$

If $x \in \mathbf{R}$, then as $g_{A+x}(z) - x = g_A(z - x)$, it follows from (11.1) that

$$g_A(z) = g_{A+x}(z + x) - x = z + \frac{\mathrm{hcap}(A)}{z + x} + \mathrm{hcap}(A)\mathrm{rad}(A + x)O\left(\frac{1}{|z + x|^2}\right). \tag{11.2}$$

Fix $\epsilon > 0$. Note that $\mathrm{hcap}(g_t(A_{t+\epsilon} \setminus A_t)) = 2\epsilon$. For $0 \leq s \leq t$, let $g_{s,t} = g_t \circ g_s^{-1}$. Applying (11.2) with $A = g_t(A_{t+\epsilon} \setminus A_t)$ and $x = -U_t$ and using that $\mathrm{rad}(g_t(A_{t+\epsilon} \setminus A_t) - U_t) \leq \mathrm{diam}(g_t(A_{t+\epsilon} \setminus A_t))$, we thus see that

$$g_{t,t+\epsilon}(z) = z + \frac{2\epsilon}{z - U_t} + 2\epsilon\mathrm{diam}(g_t(A_{t+\epsilon} \setminus A_t))O\left(\frac{1}{|z - U_t|^2}\right).$$

We thus have that

$$\begin{aligned} g_{t+\epsilon}(z) - g_t(z) &= (g_{t,t+\epsilon} - g_{t,t}) \circ g_t(z) \\ &= \frac{2\epsilon}{g_t(z) - U_t} + 2\epsilon\mathrm{diam}(g_t(A_{t+\epsilon} \setminus A_t))O\left(\frac{1}{|g_t(z) - U_t|^2}\right) \end{aligned}$$

Dividing both sides by ϵ, sending $\epsilon \to 0$, and using that (A_t) is locally growing, we thus see that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}$$

as desired. □

Theorem 11.1 implies that we can encode a family (A_t) in $\mathcal{A}$ with $A_0 = \emptyset$ in terms of a continuous, real-valued function U.

Conversely, if U is a continuous, real-valued function and we let

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

then A_t given by the complement in $\mathbf{H}$ of the domain of g_t is a family in $\mathcal{A}$ with $A_0 = \emptyset$.

The function U is called the "Loewner driving function" for (A_t).

12. Derivation of the Schramm-Loewner evolution

The purpose of this section is to explain the derivation and definition of SLE.

Definition 12.1. Suppose that (A_t) is a random family in $\mathcal{A}$ encoded with the Loewner driving function U. We say that (A_t) satisfies the *conformal Markov property* if the following is true. For each $t \geq 0$, let $\mathcal{F}_t = \sigma(U_s : s \leq t)$. Then:

(i) The conditional law of $(g_t(A_{t+s}) - U_t)_{s\geq 0}$ given $\mathcal{F}_t$ is equal to that of $(A_s)_{s\geq 0}$. (Markov property)
(ii) For each $r > 0$, $(rA_{t/r^2}) \stackrel{d}{=} (A_t)$. (Scale invariance)

Note that (i) is equivalent to the statement that, given $\mathcal{F}_t$, $(U_{t+s} - U_t)_{s\geq 0}$ has the same distribution as $(U_s)_{s\geq 0}$. That is, U has stationary, independent increments. As U is continuous, this implies that there exists $\kappa \geq 0$ and $a \in \mathbf{R}$ such that $U_t = \sqrt{\kappa} B_t + at$ where B is a standard Brownian motion.

By (ii), we have for $r > 0$ that

$$rU_{t/r^2} = \sqrt{\kappa} r B_{t/r^2} + ra(t/r^2) = \sqrt{\kappa}\widetilde{B} + at/r \stackrel{d}{=} U_t$$

where $\widetilde{B}$ is a standard Brownian motion. The only way that this can be the case is if $a = 0$.

Combining, we have just obtained Schramm's theorem.

Theorem 12.2 (Schramm)**.** *If (A_t) satisfies the conformal Markov property, then there exists $\kappa \geq 0$ such that $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion.*

For $\kappa > 0$, SLE_κ is the random family of hulls (A_t) which are obtained by solving the Loewner equation with $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion.

SLE_0 corresponds to the case $U_t \equiv 0$ for all $t \geq 0$, which corresponds to the curve $A_t = [0, 2\sqrt{t} i]$.

Remark 12.3. (i) It turns out that SLE_κ is *generated* by a continuous curve γ. That is, $\mathbf{H} \setminus A_t$ is equal to the unbounded component of $\mathbf{H} \setminus \gamma([0,t])$ for each $t \geq 0$. Equivalently, A_t is equal to the set obtained by "filling in" the holes cut off from ∞ by $\gamma|_{[0,t]}$. This result was first proved by Rohde-Schramm. In the rest of this course, we will take it as an assumption.
(ii) The behavior of SLE_κ depends strongly on κ. We will show later that SLE_κ is simple for $\kappa \in (0,4]$, self-intersecting for $\kappa \in (4,8)$, and space-filling for $\kappa \geq 8$.
(iii) As we proved just above, SLE_κ is singled out by the conformal Markov property. This is motivated from conjectures in the physics literature which regarding the behavior of scaling limits of discrete models in two dimensions (percolation, loop-erased random walk, etc...)
(iv) The main tool to analyze SLE_κ is stochastic calculus, which we will review next.

13. Stochastic calculus review

The general setting that we shall have in mind is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $(\mathcal{F}_t)$ which satisfies the *usual conditions*:

(i) $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets
(ii) $(\mathcal{F}_t)$ is right-continuous, i.e., $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$ for all $t \geq 0$.

The basic object in stochastic calculus is the *continuous semi-martingale*. This is a process X_t which can be written as a sum $M_t + A_t$ where M_t is a continuous local martingale and A_t is a process of bounded variation.

The following concepts from stochastic calculus will be important for this course:

- The stochastic integral
- The quadratic variation
- Itô's fomrula
- Lévy characterization of Brownian motion
- Stochastic differential equations

13.1. The stochastic integral.

The stochastic integral of a previsible process H_t against a semi-martingale $X_t = M_t + A_t$ is defined by setting

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dX_s.$$

The first integral on the right hand is an Itô integral and is a continuous local martingale. The second integral is a Lebesgue-Stieljes integral and is a process of bounded variation. The Itô integral is defined and constructed in a way which is similar to the Riemann integral. It exists due to the cancellation which arises since M_t is a continuous local martingale, even though M_t does not have finite variation.

13.2. Quadratic variation.

The quadratic variation of a continuous local martingale M is

$$[M]_t = \lim_{n\to\infty} \sum_{k=0}^{\lceil 2^n t\rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2.$$

It is the unique non-decreasing continuous process such that

$$M_t^2 - [M]_t$$

is a continuous local martingale. The quadratic variation of a continuous process of finite variation vanishes. So,

$$[X]_t = [M + A]_t = [M]_t.$$

Also,

$$[\int H_s dM_s]_t = \int_0^t H_s^2 d[M]_s.$$

13.3. **Itô's formula.** Itô's formula is the stochastic calculus analog of the fundamental theorem of calculus. To motiviate it, suppose that $f \in C^(\mathbf{R})$. If $t \geq 0$ and $0 = t_0 < \cdots < t_n = t$ is a partition of $[0,t]$, then we can write

$$\begin{aligned} f(t) &= f(0) + \sum_{k=1}^n \big(f(t_k) - f(t_{k-1})\big) \\ &= f(0) + \sum_{k=1}^n \big(f'(t_{k-1})(t_k - t_{k-1}) + o(t_k - t_{k-1})\big) \quad \text{(Taylor's theorem)} \\ &\to f(0) + \int_0^t f'(s) ds \quad \text{as} \quad \max_{1 \leq k \leq n} (t_k - t_{k-1}) \to 0. \end{aligned}$$

Now suppose that B is a standard Brownian motion with $B_0 = 0$. Then we can write

$$\begin{aligned} f(B_t) &= f(0) + \sum_{k=1}^n \big(f(B_{t_k}) - f(B_{t_{k-1}})\big) \\ &= f(0) + \sum_{k=1}^n \left(f'(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) + \frac{1}{2} f''(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2 + o((B_{t_k} - B_{t_{k-1}})^2)\right) \\ &\to f(0) + \int_0^t f'(B_s) dB_s + \frac{1}{2}\int_0^t f''(B_s) ds \quad \text{as} \quad \max_{1 \leq k \leq n} (t_k - t_{k-1}) \to 0. \end{aligned}$$

We have derived a special case of Itô's formula:

$$f(B_t) = f(0) + \int_0^t f'(B_s) dB_s + \frac{1}{2}\int_0^t f''(B_s) ds.$$

Here is a more general version. Suppose that $f \in C^{1,2}(\mathbf{R}_+ \times \mathbf{R})$. The first variable is the time variable and the second variable is the spatial variable. If $X_t = M_t + A_t$ is a continuous semimartingale, then Itô's formula states that:

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_s f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) dX_s + \frac{1}{2}\int_0^t \partial_x^2 f(s, X_s) d[M]_s.$$

We can rewrite this as:

$$\begin{aligned} f(t, X_t) = & f(0, X_0) + \int_0^t \partial_x f(s, X_s) dM_s + \bigg(\int_0^t \partial_s f(s, X_s) ds + \\ & \int_0^t \partial_x f(s, X_s) dA_s + \frac{1}{2}\int_0^t \partial_x^2 f(s, X_s) d[M]_s\bigg). \end{aligned}$$

The first integral is the martingale part of the semimartinagle decomposition of $f(t, X_t)$ and the other integrals together are the bounded variation part.

13.4. **Lévy characterization.** Suppose that M is a continuous local martingale. The Lévy characterization of Brownian motion states that M is a Brownian motion if and only if $[M]_t = t$ for all $t \geq 0$. It is proved by using Itô's formula to show that the process $e^{i\theta M_t + \theta^2/2[M]_t}$ is a continuous martingale.

13.5. **Stochastic differential equations.** Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ together with $(\mathcal{F}_t)$ is a probability space satisfying the usual conditions. Let B be a standard Brownian motion which is adapted to $(\mathcal{F}_t)$. If b, σ are measurable functions, then we say that a continuous semimartingale X_t adapted to $(\mathcal{F}_t)$ satisfies the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

provided

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \quad \text{for all} \quad t \geq 0.$$

It will be proved in Stochastic Calculus that this SDE has a unique solution when b, σ are Lipschitz functions.

14. Phases of SLE

Suppose that $X = (B^1, \ldots, B^d)$ is a d-dimensional Brownian motion. In other words, $B^1, \ldots, B^d$ are independent standard Brownian motions. Let

$$Z_t = \|X_t\|^2 = (B_t^1)^2 + \cdots + (B_t^d)^2.$$

By Itô's formula, we have that

$$Z_t = (B_t^1)^2 + \cdots + (B_t^d)^2 = Z_0 + 2\int_0^t B_s^1 dB_s^1 + \cdots + 2\int_0^t B_s^d dB_s^d + dt.$$

Let

$$Y_t = \int_0^t \frac{1}{Z_s^{1/2}} B_s^1 dB_s^1 + \cdots + \int_0^t \frac{1}{Z_s^{1/2}} B_s^d dB_s^d.$$

Then Y_t is a continuous local martingale with

$$\begin{aligned}
[Y]_t &= \left[\int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^1 dB_s^1 + \cdots + \int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^d dB_s^d\right]_t \\
&= \left[\int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^1 dB_s^1\right]_t + \cdots + \left[\int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^d dB_s^d\right]_t \\
&= \int_0^t \frac{1}{Z_s}(B_s^1)^2 ds + \cdots + \int_0^t \frac{1}{Z_s}(B_s^d)^2 ds \\
&= t.
\end{aligned}$$

Consequently, the Lévy characterization implies that $Y_t = \widetilde{B}_t$ where $\widetilde{B}$ is a standard Brownian motion. This allows us to write

$$Z_t = Z_0 + 2\int_0^t Z_s^{1/2} d\widetilde{B}_s + dt.$$

Equivalently,

$$dZ_t = 2Z_t^{1/2} d\widetilde{B}_t + d \cdot dt.$$

This it the "square Bessel SDE of dimension d" and we say that Z is a square Bessel process of dimension d. Sometimes, this is written as $Z_t \sim \mathrm{BESQ}^d$. This SDE has a solution for every $d \in \mathbf{R}$ which is defined at least up until the first time that the process hits 0. In particular, d need not be an integer.

By applying Itô's formula with $f(x) = \sqrt{x}$, we next see that

$$\begin{aligned} Z_t^{1/2} &= Z_0^{1/2} + \frac{1}{2}\int_0^t Z_s^{-1/2} dZ_s - \frac{1}{8}\int_0^t Z_s^{-3/2} d[Z]_s \\ &= Z_0^{1/2} + \widetilde{B}_t + \frac{d}{2}\int_0^t Z_s^{-1/2} ds - \frac{1}{2}\int_0^t Z_s^{-1/2} ds \\ &= Z_0^{1/2} + \left(\frac{d-1}{2}\right)\int_0^t Z_s^{-1/2} ds + \widetilde{B}_t. \end{aligned}$$

Thus $U_t = Z_t^{1/2}$ satisfies

$$U_t = U_0 + \left(\frac{d-1}{2}\right)\int_0^t \frac{1}{U_s} ds + \widetilde{B}_t.$$

Equivalently,

$$dU_t = \left(\frac{d-1}{2}\right)\frac{1}{U_t} dt + d\widetilde{B}_t.$$

This is the "Bessel SDE of dimension d" and we say that U is a Bessel process of dimension d. Sometimes this is written as $U_t \sim \mathrm{BES}^d$. As in the case of the square Bessel SDE, the Bessel SDE has a solution for every $d \in \mathbf{R}$ which is defined at least up until the first time that the process hits 0. So, as before, d need not be an integer.

Proposition 14.1. *Suppose that $d \in \mathbf{R}$ and $U_t \sim \mathrm{BES}^d$.*

(i) If $d < 2$, then U_t hits 0 a.s.
(ii) If $d \geq 2$, then U_t does not hit 0 a.s.

Proof. We will prove the proposition by considering the process U_t^{2-d}. By Itô's formula, we have that

$$\begin{aligned} U_t^{2-d} &= U_0^{2-d} + \int_0^t (2-d)U_s^{1-d} dU_s + \frac{1}{2}\int_0^t (2-d)(1-d)U_s^{-d} d[U]_s \\ &= U_0^{2-d} + \int_0^t (2-d)U_s^{1-d} d\widetilde{B}_s + \int_0^t \frac{(d-2)(d-1)}{2U_s} U_s^{1-d} ds + \frac{1}{2}\int_0^t (2-d)(1-d)U_s^{-d} ds \\ &= U_0^{2-d} + \int_0^t (2-d)U_s^{1-d} d\widetilde{B}_s. \end{aligned}$$

This proves that U_t^{2-d} is a continuous, local martingale. For each $a \in \mathbf{R}$, we let $\tau_a = \inf\{t \geq 0 : U_t = a\}$. If $0 \leq a < U_0 < b < \infty$, then the process $U_{t \wedge \tau_a \wedge \tau_b}^{2-d}$ is a bounded, continuous martingale. The optional stopping theorem thus implies that

$$U_0^{2-d} = \mathbf{E}[U_{\tau_a \wedge \tau_b}^{2-d}] = a^{2-d}\mathbf{P}[\tau_a < \tau_b] + b^{2-d}\mathbf{P}[\tau_b < \tau_a].$$

If $d < 2$, then we can take $a = 0$ to see that

$$U_0^{2-d} = b^{2-d}\mathbf{P}[\tau_b < \tau_0].$$

That is,

$$\mathbf{P}[\tau_b < \tau_0] = \left(\frac{U_0}{b}\right)^{2-d}.$$

By sending $b \to \infty$, we see that $\mathbf{P}[\tau_0 < \infty] = 1$. If $d > 2$, then we can write

$$\mathbf{P}[\tau_a < \tau_b] = \left(\frac{U_0}{a}\right)^{2-d} - \left(\frac{b}{a}\right)^{2-d}\mathbf{P}[\tau_b < \tau_a].$$

Taking a limit as $a \to 0$, we see that $\mathbf{P}[\tau_0 < \tau_b] = 0$ for any b. Therefore $\mathbf{P}[\tau_0 < \infty] = 0$. The case $d = 2$ is proved similarly but with $\log U_t$ in place of U_t^{2-d}. □

Suppose that (g_t) solves the chordal Loewner equation driven by $U_t = \sqrt{\kappa}B_t$ where B is a standard Brownian motion. That is,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Let γ be the curve which corresponds to the family of hulls encoded by (g_t). For each $x \in \mathbf{R}$, let $V_t^x = g_t(x) - U_t$ and let $\tau_x = \inf\{t \geq 0 : V_t^x = 0\}$. Then τ_x is the first time that x is cut off from ∞ by γ. Note that

$$dV_t^x = \frac{2}{g_t(x) - U_t}dt - dU_t = \frac{2}{V_t^x}dt - \sqrt{\kappa}dB_t.$$

Equivalently,

$$d(V_t^x/\sqrt{\kappa}) = \frac{2/\kappa}{V_t^x/\sqrt{\kappa}}dt + d\widetilde{B}_t \quad \text{where} \quad \widetilde{B}_t = -B_t.$$

That is, $V_t^x/\sqrt{\kappa}$ is a BES^d with

$$\frac{d-1}{2} = \frac{2}{\kappa}$$

hence

$$d = 1 + \frac{4}{\kappa}.$$

Note that $d \geq 2$ if and only if $\kappa \leq 4$. Consequently, $\tau_x < \infty$ if and only if $\kappa > 4$.

Proposition 14.2. SLE_κ *corresponds to a simple curve for* $\kappa \leq 4$. *It is self-intersecting for* $\kappa > 4$.

Proof. The above considerations imply that SLE_κ intersects $\partial\mathbf{H}$ if and only if $\kappa > 4$. Suppose that $t > 0$ is fixed. Then $s \mapsto g_t(\gamma(s+t)) - U_t$ is an SLE_κ curve. The proposition follows as, for each $t \geq 0$, intersection points between $\gamma|_{[t,\infty)}$ and $\gamma|_{[0,t]}$ correspond to points where the curve $s \mapsto g_t(\gamma(s+t)) - U_t$ hits the boundary. □

We are now going to show that SLE_κ for $\kappa \in (4, 8)$ cuts off regions from ∞ and that SLE_κ for $\kappa \geq 8$ fills the boundary and does not cut off regions from ∞. It will be shown on Example Sheet 2 that SLE_κ for $\kappa \geq 8$ in fact fills all $\mathbf{H}$ (i.e., is space-filling).

For the rest of this section, we will assume that $\kappa > 4$.

To this end, for $0 < x < y$, we let $g(x,y) = \mathbf{P}[\tau_x = \tau_y]$ be the probability that both x and y are cut off from ∞ at the same time. We make two observations about $g(x,y)$:

- $g(x,y) = g(1, y/x)$ since SLE_κ is scale-invariant.
- $g(1,r) \to 0$ as $r \to \infty$ since $\mathbf{P}[\tau_1 < t] \to 1$ as $t \to \infty$ and $\mathbf{P}[\tau_r < t] \to 0$ as $r \to \infty$ with t fixed.

We say that events A, B are *equivalent* if $\mathbf{P}[A \setminus B] = \mathbf{P}[B \setminus A] = 0$, i.e., A, B differ by an event of probability 0.

Lemma 14.3. *Fix $r > 1$. The event $\{\tau_1 = \tau_r\}$ is equivalent to the event*

$$E = \left\{ \sup_{t < \tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty \right\}.$$

Proof. Indeed, if E occurs then we cannot have that $\tau_1 < \tau_r$. Therefore $E \subseteq \{\tau_1 = \tau_r\}$. On the other hand, if $M > 0$, then we have that

$$\mathbf{P}\left[\tau_1 = \tau_r \,|\, \sup_{t < \tau_1} \frac{V_t^r - V_t^1}{V_t^1} \geq M \right] = \mathbf{P}[\tau_1 = \tau_r \,|\, \sigma_M < \tau_1]$$

where $\sigma_M = \inf\{t \geq 0 : (V_t^r - V_t^1)/V_t^1 \geq M\}$. By the scale-invariance of SLE_κ and the strong Markov property applied at the stopping time σ_M, we therefore have that

$$\mathbf{P}[\tau_1 = \tau_r \,|\, \sigma_M < \tau_1] = g(1, 1+M) \to 0 \quad \text{as} \quad M \to \infty.$$

This implies that

$$\mathbf{P}\left[\tau_1 = \tau_r, E^c\right] = 0,$$

which concludes the proof that $\{\tau_1 = \tau_r\}$ and E are equivalent. □

Our goal now is to show that

$$\mathbf{P}[\sup_{t < \tau_1} (V_t^r - V_t^1)/V_t^1 < \infty]$$

is positive if $\kappa \in (4,8)$ and is equal to 0 if $\kappa \geq 8$. Let

$$Z_t = \log\left(\frac{V_t^r - V_t^1}{V_t^1} \right).$$

With $d = 1 + 4/\kappa$, we have by Itô's formula that

$$dZ_t = \left(\left(\frac{3}{2} - d \right) \frac{1}{(V_t^1)^2} + \left(\frac{d-1}{2} \right) \left(\frac{V_t^r - V_t^1}{(V_t^1)^2 V_t^r} \right) \right) dt - \frac{1}{V_t^1} dB_t \quad \text{with} \quad Z_0 = \log(r-1).$$

We are now going to perform a time-change to turn the local martingale part of Z_t into a standard Brownian motion. Let

$$\sigma(t) = \inf\left\{ u \geq 0 : \int_0^u \frac{1}{(V_s^1)^2} ds = t \right\}.$$

Then we have that

$$t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} ds \quad \text{hence} \quad dt = \frac{d\sigma(t)}{(V_{\sigma(t)}^1)^2}.$$

Note that the process

$$\widetilde{B}_t = -\int_0^{\sigma(t)} \frac{1}{V_s^1} dB_s$$

is a continuous local martingale with

$$[\widetilde{B}]_t = \left[-\int_0^{\sigma(\cdot)} \frac{1}{V_s^1} dB_s \right]_t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} ds = t.$$

Therefore the Lévy characterization implies that $\widetilde{B}$ is a standard Brownian motion. Thus, with $\widetilde{Z}_t = Z_{\sigma(t)}$, we have that

$$d\widetilde{Z}_t = \left(\left(\frac{3}{2} - d \right) + \left(\frac{d-1}{2} \right) \left(\frac{V_{\sigma(t)}^r - V_{\sigma(t)}^1}{V_{\sigma(t)}^r} \right) \right) dt + d\widetilde{B}_t.$$

Consequently,

$$\begin{aligned}
\widetilde{Z}_t &= \widetilde{Z}_0 + \widetilde{B}_t + \left(\frac{3}{2} - d \right) t + \frac{d-1}{2} \int_0^t \frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} ds \\
&\geq \widetilde{Z}_0 + \widetilde{B}_t + \left(\frac{3}{2} - d \right) t.
\end{aligned}$$

If $\kappa \geq 8$ then $d = 1 + 4/\kappa \leq 3/2$, in which case we have that

$$\widetilde{Z}_t \geq \widetilde{Z}_0 + \widetilde{B}_t.$$

Hence

$$\sup_{t \geq 0} \widetilde{Z}_t \geq \widetilde{Z}_0 + \sup_{t \geq 0} \widetilde{B}_t = \infty.$$

As $\sigma(\infty) = \tau_1$, we thus have that

$$\sup_{t < \tau_1} e^{Z_t} = \infty.$$

We conclude that $g(x, y) = 0$ for all $0 < x < y$ if $\kappa \geq 8$. We have just established the following.

Proposition 14.4. *An* SLE_κ *for* $\kappa \geq 8$ *almost surely fills* $\partial \mathbf{H}$. *In particular, such a process does not cut regions off from* ∞.

Now suppose that $\kappa \in (4, 8)$. Fix $\epsilon > 0$ and assume that $r = 1 + \epsilon/2$. Note $\widetilde{Z}_0 = \log(r-1) = \log(\epsilon/2)$. Let

$$\tau = \inf\{t \geq 0 : \widetilde{Z}_t = \log \epsilon\}.$$

Then

$$\begin{aligned}
\widetilde{Z}_{t \wedge \tau} &= \widetilde{Z}_0 + \widetilde{B}_{t \wedge \tau} + \left(\frac{3}{2} - d \right) t \wedge \tau + \left(\frac{d-1}{2} \right) \int_0^{t \wedge \tau} \frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} ds \\
&\leq \widetilde{Z}_0 + \widetilde{B}_{t \wedge \tau} + \left(\frac{3}{2} - d \right) t \wedge \tau + \left(\frac{d-1}{2} \right) \int_0^{t \wedge \tau} e^{\widetilde{Z}_s} ds
\end{aligned}$$

$$\leq \widetilde{Z}_0 + \widetilde{B}_{t\wedge\tau}\left(\left(\frac{3}{2}-d\right)+\left(\frac{d-1}{2}\right)\epsilon\right)t\wedge\tau$$
$$= \widetilde{Z}_0 + \widetilde{B}_{t\wedge\tau} + at\wedge\tau \quad \text{where} \quad a = \left(\frac{3}{2}-d\right)+\left(\frac{d-1}{2}\right)\epsilon.$$

Assume that $\epsilon > 0$ is taken to be sufficiently small so that $a < 0$ (recall that $d > 3/2$ since $\kappa \in (4,8)$). Let

$$Z_t^* = \widetilde{Z}_0 + \widetilde{B}_t + at.$$

Then

$$Z_{t\wedge\tau}^* \geq \widetilde{Z}_{t\wedge\tau}.$$

As Z_t^* is a Brownian motion with negative drift starting from $\log(\epsilon/2)$, it follows that

$$\mathbf{P}[\sup_{t\geq 0} Z_t^* < \log\epsilon] > 0.$$

Therefore

$$\mathbf{P}[\sup_{t\geq 0} \widetilde{Z}_t < \log\epsilon] > 0.$$

Hence

$$\mathbf{P}\left[\sup_{t<\tau_1} e^{Z_t} < \epsilon\right] > 0.$$

This implies that $g(1, 1+\epsilon/2) > 0$. It follows from the scale-invariance and Markov property for SLE_κ that then $g(x,y) > 0$ for all $0 < x < y$ as desired (see Example Sheet 2). We have just established the following:

Proposition 14.5. *An* SLE_κ *for* $\kappa \in (4,8)$ *almost surely cuts off regions from* ∞*.*

15. Locality of SLE_6

So far, we have only defined SLE_κ in $\mathbf{H}$ from 0 to ∞. If $D \subseteq \mathbf{C}$ is a simply connected domain and $x, y \in \partial D$ are distinct, then there exists a conformal transformation $\phi\colon \mathbf{H} \to D$ with $\phi(0) = x$ and $\phi(\infty) = y$. An SLE_κ γ in D from x to y is defined by taking it to be $\phi(\widetilde{\gamma})$ where $\widetilde{\gamma}$ is an SLE_κ in $\mathbf{H}$ from 0 to ∞. (It will be shown on Example Sheet 3 that this definition is well-defined.)

We will now analyze the question of which SLE_κ should correspond to the scaling limit of percolation.

Suppose that $D \subseteq \mathbf{C}$ is simply connected, $x, y \in \partial D$ are distinct. Consider $p = 1/2$ (critical) percolation on the hexagonal lattice with hexagons of size ϵ which intersect $\overline{D}$. We take the hexagons which intersect the clockwise (resp. counterclockwise) segment of ∂D from x to y to be all black (resp. white). With this choice of boundary conditions, there exists a unique interface γ^ϵ which connects x to y with black (resp. white) hexagons on its left (resp. right) side. (See Figure 15.1 for an illustration and Figure 1.3 for actual simulations in the special case of a lozenge shaped domain.)

It was conjectured (now proved by Smirnov) that the limit γ of γ^ϵ is *conformally invariant*. This means that if $\widetilde{D}$ is another simply connected domain, $\widetilde{x}, \widetilde{y} \in \partial\widetilde{D}$ are distinct, and $\psi\colon D \to \widetilde{D}$ is a

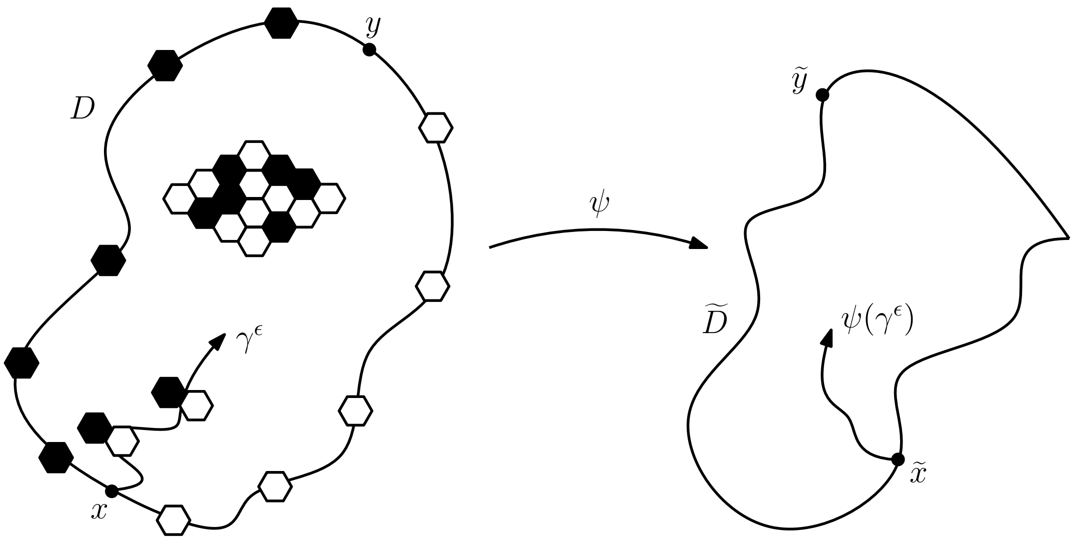

FIGURE 15.1. Percolation exploration γ^ϵ in the hexagonal lattice with hexagons of size ϵ in a simply connected domain D from x to y with black (resp. white) hexagons on the clockwise (resp. counterclockwise) arc of ∂D from x to y. Some representative hexagons are shown together with their colors. The scaling limit γ of γ^ϵ as $\epsilon \to 0$ was conjectured (now proved by Smirnov) to be conformally invariant, which means that if $\psi\colon D \to \widetilde{D}$ is a conformal transformation with $\widetilde{x} = \psi(x)$ and $\widetilde{y} = \psi(y)$, then the law of $\psi(\gamma) = \lim_{\epsilon\to 0} \psi(\gamma^\epsilon)$ is equal in distribution to the scaling limit of the percolation exploration in $\widetilde{D}$ from $\widetilde{x}$ to $\widetilde{y}$ with the corresponding black/white boundary conditions.

conformal transformation, then $\psi(\gamma)$ is equal in distribution to the scaling limit of percolation on $\widetilde{D}$ from $\widetilde{x}$ to $\widetilde{y}$ with boundary conditions analogous to those described just above. (See Figure 15.1 for an illustration.)

Also, percolation satisfies a natural Markov property (this is its spatial Markov property). Namely, if you condition on γ^ϵ up to a time t, then the conditional law of the remainder of the percolation interface is that of a percolation exploration in the remaining domain from $\gamma^\epsilon(t)$ to y. The reason for this is that in order to observe γ^ϵ, one need only observe the black (resp. white) hexagons which are on its left (resp. right) side.

If the scaling limit γ of the percolation exploration exists and it is conformally invariant, then the above considerations imply that it must satisfy the conformal Markov property. Therefore there must exist $\kappa \geq 0$ such that γ is an SLE_κ. We will now show that the only κ value which can correspond to the scaling limit of percolation is $\kappa = 6$.

Percolation possesses the extra property which is referred to as "locality". In special situation that we consider the percolation exploration on $\mathbf{H}$, it can be formulated as follows (but is indeed a very general principle). Suppose that D is a simply connected domain in $\mathbf{H}$ with 0 on its boundary. Then a percolation exploration in D with black (resp. white) boundary conditions on $\mathbf{R}_- \cap \partial D$ (resp. $\mathbf{R}_+ \cap \partial D$), run up until hitting $\partial D \setminus \partial \mathbf{H}$, has the same distribution as a percolation exploration in

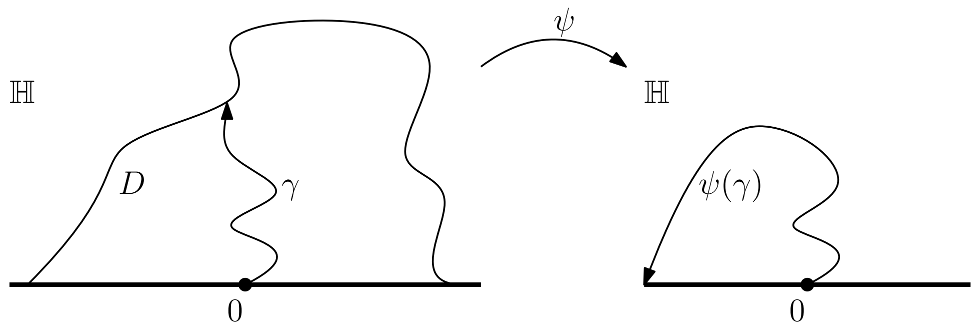

FIGURE 15.2. Illustration of the locality property for SLE_κ. Shown on the left is an SLE_κ curve γ in $\mathbf{H}$ from 0 to ∞ stopped upon leaving a simply connected domain $D \subseteq \mathbf{H}$ with $0 \in \partial D$. SLE_κ is said to satisfy the locality property if γ has the same distribution as an SLE_κ in D, stopping upon hitting $\partial D \setminus \partial \mathbf{H}$. Equivalently, if ψ is a conformal transformation $D \to \mathbf{H}$ fixing 0, then $\psi(\gamma)$ has the law of an SLE_κ in $\mathbf{H}$ from 0 to ∞, stopped upon hitting $\psi(\partial D \setminus \partial \mathbf{H})$. It turns out that locality holds if and only if $\kappa = 6$, which implies that the only SLE_κ which can correspond to the scaling limit of percolation is SLE_6.

all of $\mathbf{H}$ with black (resp. white) boundary conditions on $\mathbf{R}_-$ (resp. $\mathbf{R}_+$), also stopped upon hitting $\partial D \setminus \partial \mathbf{H}$.

Therefore, the corresponding SLE_κ should satisfy an analogous property. That is, we want to figure out for which value of κ the following is true. Suppose that $D \subseteq \mathbf{H}$ is a simply connected domain with 0 on its boundary. Let γ be an SLE_κ in $\mathbf{H}$ from 0 and consider γ stopped upon hitting $\partial D \setminus \partial \mathbf{H}$. Then we want that γ has the same law as an SLE_κ in D starting from 0 stopped at the analogous time. Equivalently, if $\psi\colon D \to \mathbf{H}$ is a conformal transformation with $\psi(0) = 0$, then we want that $\psi(\gamma)$ is an SLE_κ in $\mathbf{H}$. This is the so-called "locality property."

We will now show that locality holds if and only if $\kappa = 6$.

In order to establish this, we need to understand how the Loewner evolution changes when we apply a conformal transformation. Suppose that (A_t) is a non-decreasing family of compact $\mathbf{H}$-hulls which are locally growing and are parameterized by capacity and assume that $T > 0$ is such that $A_T \subseteq D$. For each $t \in [0, T]$, let $\widetilde{A}_t = \psi(A_t)$. Then $(\widetilde{A})_{t \in [0,T]}$ is a family of compact $\mathbf{H}$-hulls which are non-decreasing, locally growing, and with $\widetilde{A}_0 = \emptyset$.

For each $t \geq 0$, let $\widetilde{g}_t = g_{\widetilde{A}_t}$ be the unique conformal transformation $\mathbf{H} \setminus \widetilde{A}_t \to \mathbf{H}$ with $\widetilde{g}_t(z) - z \to 0$ as $z \to \infty$. Let $\widetilde{a}(t) = \text{hcap}(\widetilde{A}_t)$. It will be shown on Example Sheet 3 that $(\widetilde{g}_t)$ satisfies

$$\partial_t \widetilde{g}_t(z) = \frac{\partial_t \widetilde{a}(t)}{\widetilde{g}_t(z) - \widetilde{U}_t}, \quad \widetilde{g}_0(z) = z \tag{15.1}$$

where $\widetilde{U}_t = \psi_t(U_t)$ for $\psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1}$, (g_t) the Loewner evolution associated with (A_t), and U_t its Loewner driving function. Also, (see Example Sheet 3)

$$\widetilde{a}(t) = \int_0^t 2(\psi_s'(U_s))^2 ds. \tag{15.2}$$

(The formula (15.2) is intuitive — and indeed derived — if one recalls the scaling property for half-plane capacity deduced earlier.)

We will want to apply Itô's formula to deduce the semi-martingale form of $\widetilde{U}_t = \psi_t(U_t)$. In order to do so, we need to identify the time-derivative of ψ_t evaluated at U_t.

Proposition 15.1. *The maps* (ψ_t) *satisfy*

$$\partial_t \psi_t(z) = 2\left(\frac{(\psi_t'(U_t))^2}{\psi_t(z) - \psi_t(U_t)} - \psi_t'(z)\frac{1}{z - U_t}\right).$$

Moreover, at $z = U_t$, *we have*

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

Proof. We have that

$$\begin{aligned}\partial_t \psi_t(z) &= (\partial_t \widetilde{g}_t)(\psi(g_t^{-1}(z))) + \widetilde{g}_t'(\psi(g_t^{-1}(z)))\psi'(g_t^{-1}(z))\partial_t(g_t^{-1}(z)) \\ &= \frac{2(\psi_t'(U_t))^2}{\psi_t(z) - \psi_t(U_t)} - \psi_t'(z)\frac{2}{z - U_t}.\end{aligned}$$

This proves the first assertion of the proposition, where we have used the identity

$$0 = \partial_t(g_t^{-1}(g_t(z))) = (\partial_t g_t^{-1})(g_t(z)) + (g_t^{-1})'(g_t(z))\frac{2}{g_t(z) - U_t}$$

in order to derive the formula for $\partial_t g_t^{-1}(z)$.

The second assertion of the proposition is on Example Sheet 3. □

Suppose that $U_t = \sqrt{\kappa}B_t$ where B is a standard Brownian motion. By Itô's formula, we have that

$$\begin{aligned}dU_t &= d\psi_t(U_t) \\ &= \left(\partial_t \psi_t(U_t) + \frac{\kappa}{2}\psi_t''(U_t)\right)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t \\ &= \left(-3\psi_t''(U_t) + \frac{\kappa}{2}\psi_t''(U_t)\right)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t \quad \text{(by Proposition 15.1)} \\ &= \frac{\kappa - 6}{2}\psi_t''(U_t)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t.\end{aligned}$$

We now let

$$\sigma(t) = \inf\{u \geq 0 : \int_0^u (\psi_s'(U_s))^2 ds = t\}.$$

Then

$$\partial_t \widetilde{g}_{\sigma(t)}(z) = \frac{2}{\widetilde{g}_{\sigma(t)} - \widetilde{U}_{\sigma(t)}}dt, \quad \widetilde{g}_{\sigma(0)}(z) = z.$$

Also, if we let $\widetilde{U}_t^* = \widetilde{U}_{\sigma(t)}$, then we have that

$$d\widetilde{U}_t^* = \frac{\kappa - 6}{2}\frac{\psi_{\sigma(t)}''(U_{\sigma(t)})}{(\psi_{\sigma(t)}'(U_{\sigma(t)}))^2}dt + \sqrt{\kappa}d\widetilde{B}_t$$

where

$$\widetilde{B}_t = \int_0^{\sigma(t)} \psi_s'(U_s) dB_s$$

is a standard Brownian motion (by the Lévy characterization). In particular, if $\kappa = 6$ then we have that $\widetilde{U}_t^* = \sqrt{6}\widetilde{B}_t$. That is, $(\widetilde{A}_{\sigma(t)})$ is equal in distribution to the family of hulls associated with an SLE_6. We have now obtained the following theorem:

Theorem 15.2. *If γ is an SLE_6 curve, then $\psi(\gamma)$ is an SLE_6 (up until first hitting $\psi(\partial D \setminus \partial \mathbf{H})$ and considered modulo a time-change).*

We conclude that SLE_6 is the only possible SLE curve which could describe the scaling limit of percolation.

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