SCHRAMM-LOEWNER EVOLUTIONS

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Preface

These lecture notes are for the University of Cambridge Part III course Schramm-Loewner Evolutions, given Lent 2019. Please notify jpmiller@statslab.cam.ac.uk for corrections.

1. Introduction

The Schramm-Loewner evolution (SLE) is a random fractal curve which lives in a domain D in the complex plane $\mathbb{C}$. It was introduced by Schramm in 1999 to describe the scaling limits of interfaces in two-dimensional discrete models from statistical mechanics. It has been a transformative idea which has led to new unexpected links between a number of probabilistic models and also other areas of mathematics.

Here are three important examples where SLE's arise.

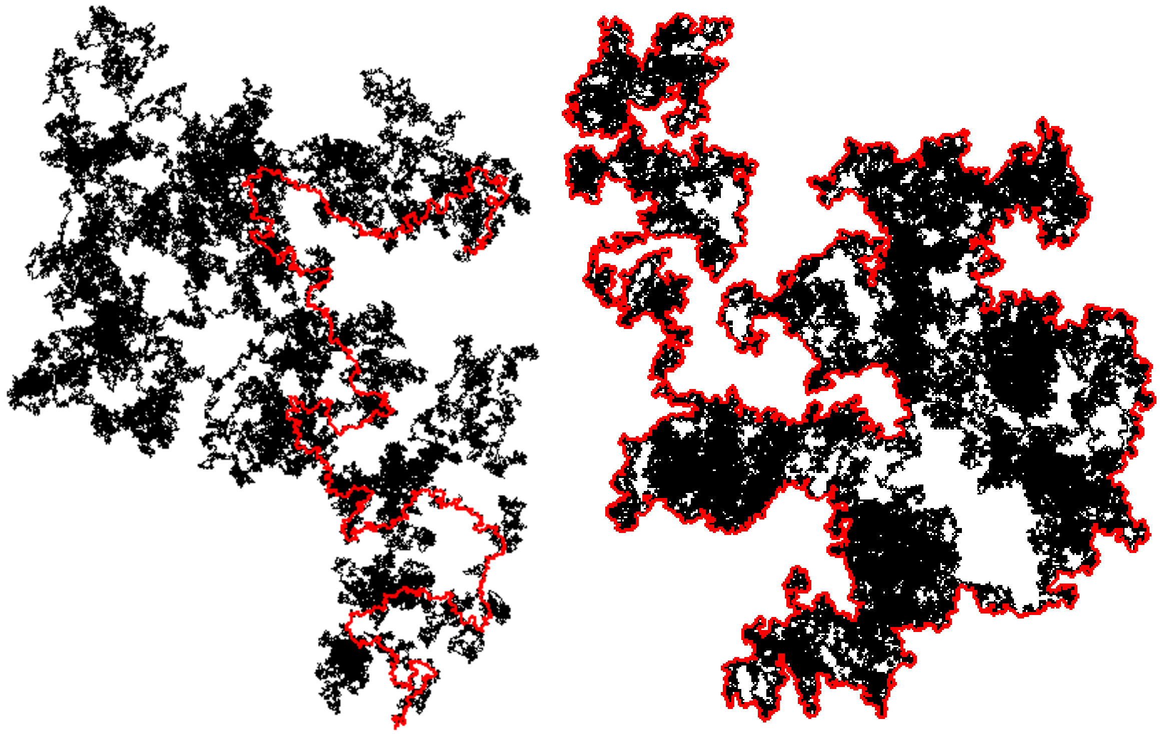

FIGURE 1.1. **Left:** A random walk (black) on $\mathbb{Z}^2$ and its loop-erasure (red). It was proved by Lawler-Schramm-Werner the scaling limit of the loop-erasure is given by an SLE_2 curve. **Right:** The range of a planar Brownian motion shown in black and its outer boundary shown in red. It was conjectured by Mandelbrot that the dimension of the outer boundary is equal to $\frac{4}{3}$. Mandelbrot's conjecture was proved by Lawler-Schramm-Werner using SLE.

Example 1.1 (Loop-erased random walk on $\mathbb{Z}^2$)**.** A (simple) random walk on $\mathbb{Z}^2$ is a particle X_n which in each time step goes up/down/left/right with equal probability 1/4. The loop-erasure of X_n is defined by erasing the loops that X_n makes chronologically. It is an important object in probability because it is connected to many other probabilistic models (e.g., uniform spanning trees, dimers, sand-pile models). See the left side of Figure 1.1 for a simulation of a long random walk together with its loop-erasure. By Donsker's invariance principle, $X_{\lfloor nt \rfloor}/\sqrt{n}$ converges in the limit to a two-dimensional Brownian motion. A natural question to ask is what continuous object describes the scaling limit of the loop-erasure of X_n. It was proved by Lawler-Schramm-Werner that it is given by an SLE_2 curve.

Example 1.2 (Outer boundary of Brownian motion)**.** Suppose that $X = (B_1, B_2)$ is a planar Brownian motion. That is, B_1, B_2 are independent standard Brownian motions. The *outer boundary* of $X([0,1])$ is the boundary of the unbounded component of $\mathbb{C} \setminus X([0,1])$. See the right side of Figure 1.1 for a simulation of a planar Brownian motion with emphasis on its outer boundary. Mandelbrot conjectured state that the Hausdorff dimension, a measure theoretic notion of dimension, is equal to $\frac{4}{3}$. This conjecture was proved by Lawler-Schramm-Werner.

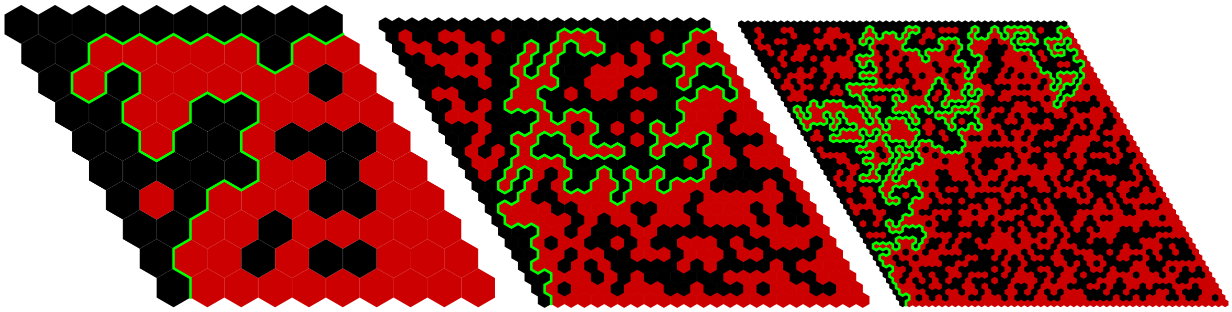

FIGURE 1.2. Critical percolation on a lozenge shaped subset of the hexagonal lattice in $\mathbb{C}$ with black boundary conditions on the left and top sides and red boundary conditions on the bottom and right sides. This choice of boundary conditions forces the existence of a unique interface (green) from the bottom corner of the lozenge to the top which has black (resp. red) hexagons on its left (resp. right) side. It was proved by Smirnov that the scaling limit of this interface converges in the limit to an SLE_6 curve. The left, middle, and right lozenges respectively have side length 10, 25, and 50.

Example 1.3 (Percolation interface)**.** Consider the hexagonal lattice in the plane. We color each hexagon either "white" or "black" independently with equal probability $\frac{1}{2}$. See Figure 1.2 for a numerical simulation. A famous question in probability for many years was to describe the large scale behavior of the interfaces between the white and the black sites. This problem was solved by Smirnov, who showed that they converge in the limit to SLE_6 curves.

Famous open question: prove the same thing for any *other* planar lattice, such as $\mathbb{Z}^2$.

The remainder of this course is structured as follows:

- We will first review the complex analysis and probability background in order to derive and define SLE.
- We will then establish some of the basic properties of SLE.
- Finally, we will describe some more recent developments in the field.

2. Conformal mapping review

Suppose that U, V are domains in $\mathbb{C}$ and that $f\colon U \to V$ is a map. We say that f is *holomorphic* if it is complex differentiable, i.e., for each $z \in U$ then limit

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \quad \text{exists.}$$

A *conformal transformation* is a map which is a bijection (also sometimes called a "conformal equivalence" or just "conformal").

A domain $U \subseteq \mathbb{C}$ is called *simply connected* if $\mathbb{C} \setminus U$ is connected. Important examples of simply connected domains include the complex plane $\mathbb{C}$, the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Theorem 2.1 (Riemann mapping theorem)**.** *Suppose that U is a simply connected domain with $U \neq \mathbb{C}$ and $z \in U$. Then there exists a unique conformal transformation $f \colon \mathbb{D} \to U$ with $f(0) = z$ and $f'(0) > 0$.*

We will not give a proof of the Riemann mapping theorem here. It can be found in most complex analysis textbooks. An immediate consequence of the Riemann mapping theorem is that any two simply connected domains which are both distinct from $\mathbb{C}$ can be mapped to each other using a conformal transformation.

Corollary 2.2. *If U, V are simply connected domains with $U, V \neq \mathbb{C}$ and $z \in U$ and $w \in V$, then there exists a unique conformal transformation $f \colon U \to V$ with $f(z) = w$ and $f'(z) > 0$.*

2.1. **Examples.** Conformal transformations of $\mathbb{D}$. Suppose that $U = \mathbb{D}$ and $z \in \mathbb{D}$. Then $f \colon \mathbb{D} \to \mathbb{D}$ given by

$$f(w) = \frac{w + z}{1 + \overline{z}w}$$

is the unique conformal transformation with $f(0) = z$ and $f'(0) > 0$. More generally, every conformal transformation $f \colon \mathbb{D} \to \mathbb{D}$ is of the form

$$f(w) = \lambda \frac{w - z}{\overline{z}w - 1}$$

where $\lambda \in \partial\mathbb{D}$ and $z \in \mathbb{D}$. So, there is a three-real-parameter family of such maps (z corresponds to two parameters and λ to one).

The map $f \colon \mathbb{H} \to \mathbb{D}$ given by

$$f(z) = \frac{z - i}{z + i}$$

is a conformal transformation. It is the so-called Cayley transform. Its inverse $g \colon \mathbb{D} \to \mathbb{H}$ is given by

$$g(w) = \frac{i(1 + w)}{1 - w}$$

and is also a conformal transformation.

The conformal transformations $\mathbb{H} \to \mathbb{H}$ consist of the maps of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

More generally, if U, V are simply connected domains with $U, V \neq \mathbb{C}$, then there is a three-parameter family of conformal transformations $f \colon U \to V$.

Here is another important example which motivates the definition of SLE. For each $t \geq 0$, let $H_t = \mathbb{H} \setminus [0, 2\sqrt{t}i]$. Let $g_t \colon H_t \to \mathbb{H}$ be the map $z \mapsto \sqrt{z^2 + 4t}$. Then g_t is a conformal transformation $H_t \to \mathbb{H}$.

We make two observations about the family of conformal maps (g_t). First, we have that

$$|g_t(z) - z| = |\sqrt{z^2 + 4t} - z| \to 0 \quad \text{as} \quad z \to \infty.$$

That is, “g_t looks like the identity map at ∞.”

Second, we have that

$$\partial_t g_t(z) = \frac{1}{2\sqrt{z^2+4t}} \cdot 4 = \frac{2}{g_t(z)}.$$

So, for each $z \in \mathbb{H}$ fixed we have that $g_t(z)$ solves the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z)}, \quad \text{with} \quad g_0(z) = z. \tag{2.1}$$

For each $z \in \mathbb{H}$, the basic existence and uniqueness theorem for ODEs implies that (2.1) has a unique solution up until the denominator on the right hand side explodes, i.e.

$$\tau(z) = \inf\{t \geq 0 : \mathrm{Im}(g_t(z)) = 0\}.$$

In other words, the family of conformal transformations (g_t) are characterized by (2.1). In particular, the curve $\gamma(t) = 2\sqrt{t}i$ is encoded by (2.1). This is a special case of Loewner's theorem.

Here is a preview for later on in the course. Suppose that γ is any simple curve (i.e., non-self-intersecting) in $\mathbb{H}$ starting from 0. For each $t \geq 0$, let g_t be the unique conformal transformation which maps $H_t := \mathbb{H} \setminus \gamma([0,t])$ to $\mathbb{H}$ with $|g_t(z) - z| \to \infty$. (We will later prove that there indeed does exist a unique such conformal transformation.) Then Loewner's theorem states that there exists a continuous, real-valued function W such that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad \text{with} \quad g_0(z) = z.$$

This is the so-called *chordal Loewner equation*. Using this equation, we see that there is a correspondence between simple curves in $\mathbb{H}$ and continuous, real-valued functions.

The case $\gamma(t) = 2\sqrt{t}i$ corresponds to $W = 0$.

SLE_κ corresponds to the case $W = \sqrt{\kappa}B$ where B is a standard Brownian motion.

3. Brownian motion, harmonic functions, and conformal maps

Recall that $f = u + iv$ is holomorphic if and only if u satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3.1}$$

One important consequence of the Cauchy-Riemann equations is that if f is holomorphic then u, v are harmonic. This means that

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \quad \text{and} \quad \Delta v = 0.$$

Indeed,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}.$$

We will now recall a few results which were proved in Advanced Probability which serve to relate harmonic functions and Brownian motion. Throughout, we say that a process $B = B^1 + iB^2$ is a *complex Brownian motion* if B^1, B^2 are independent standard Brownian motions in $\mathbb{R}$.

Theorem 3.1. *Let u be a harmonic function on a bounded domain D which is continuous on $\overline{D}$. Fix $z \in D$ and let $\mathbb{P}_z$ be the law of a complex Brownian motion B starting from z and let $\tau = \inf\{t \geq 0 : B_t \notin D\}$. Then*

$$u(z) = \mathbb{E}_z[u(B_\tau)].$$

Proof. This was proved in Advanced Probability. Another proof based on Itô's formula will be given in Stochastic Calculus. □

Theorem 3.2 (Mean-value property for harmonic functions)**.** *In the setting of the prevoius theorem if $z \in D$ and $r > 0$ are such that $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\} \subseteq D$, then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta.$$

Proof. This was proved in Advanced Probability. □

Theorem 3.3 (Maximum principle)**.** *Suppose that u is harmonic in a domain D. If u attains its maximum at an interior point in D, then u is constant.*

Proof. Assume that u attains its maximum at $z_0 \in D$. Let $D_0 = \{z \in D : u(z) = u(z_0)\}$. Then $D_0 \neq \emptyset$ since $z_0 \in D_0$. The continuity of u in D implies that D_0 is (relatively) closed in D. Suppose that $z \in D_0$ and $r > 0$ is such that $B(z,r) \subseteq D$. Then $u|_{\partial B(z,r)} = u(z_0)$ for otherwise there exists $w \in \partial B(z,r)$ and $\epsilon > 0$ such that u is at most $u(z_0) - \epsilon$ on $B(w,\epsilon)$ which, by the mean-value property, would contradict that $u(z) = u(z_0)$. Combining this with Theorem 3.1 implies that u is constant on $B(z,r)$. Therefore D_0 is open hence $D_0 = D$. □

Theorem 3.4 (Maximum modulus principle)**.** *Let D be a domain and let $f : D \to \mathbb{C}$ be a holomorphic map. If $|f|$ attains its maximum in the interior of D, then f is constant.*

Proof. Assume that f attains its maximum at $z_0 \in D$. Let K be compact in D with $z_0 \in K$. Assume further that the interior of K is connected and that K is the closure of its interior. By replacing f with $f + M$ for $M \in \mathbb{R}$ sufficiently large, we can assume that $|f| \neq 0$ on K. Note that $\log|f|$ is a harmonic function on K. As $|f|$ attains its maximum in D on K, it follows that $\log|f|$ does as well, hence $\log|f|$ is constant on K by the maximum principle. Therefore $|f|$ is constant on K as well. Since K was an arbitrary compact subset of D containing z_0 (which is connected and is the closure of its interior), we deduce that $|f|$ is constant on all of D. This implies that $f(D)$ is contained in a circle in $\mathbb{C}$ hence the Lebesgue measure of $f(D)$ is equal to 0. It is easy to see that if $f'(z) \neq 0$ for some $z \in D$, then the area of $f(D)$ is strictly positive. Therefore $f'(z) = 0$ for all $z \in D$, which implies that f is constant on D. □

Theorem 3.5 (Schwarz Lemma)**.** *Suppose that $f\colon \mathbb{D}\to\mathbb{D}$ is a holomorphic map with $f(0)=0$. Then $|f(z)|\le|z|$ for all $z\in\mathbb{D}$. If $|f(z)|=|z|$ for some $z\in\mathbb{D}$, then there exists $\theta\in\mathbb{R}$ so that $f(w)=we^{i\theta}$ (i.e., f is a rotation map).*

Proof. Let

$$g(z)=\begin{cases} f(z)/z & \text{if} \quad z\neq 0,\\ f'(0) & \text{if} \quad z=0.\end{cases}$$

Then g is a holomorphic map on $\mathbb{D}$ and $|g(z)|\le 1$ for all $z\in\mathbb{D}$ by the maximum modulus principle. If $|f(z_0)|=|z_0|$ for some $z_0\in\mathbb{D}\setminus\{0\}$ then the maximum modulus principle implies that there exists $c\in\mathbb{C}$ such that $g(z)=c$ for all $z\in\mathbb{D}$. As $|g(z_0)|=1$ it follows that $|c|=1$. That is, there exists $\theta\in\mathbb{R}$ so that $c=e^{i\theta}$. Hence, $f(w)=e^{i\theta}w$ as claimed. □

4. Distortion estimates for conformal maps

Let $\mathcal{U}$ be the collection of conformal transformations $f\colon\mathbb{D}\to D$, where D is any simply connected domain with $0\in D$ and $D\neq\mathbb{C}$, with $f(0)=0$ and $f'(0)=1$. Note that if $f\in\mathcal{U}$ then

$$f(z)=\sum_{n=0}^{\infty}a_nz^n=z+\sum_{n=2}^{\infty}a_nz^n.$$

Theorem 4.1 (Koebe-1/4 theorem)**.** *If $f\in\mathcal{U}$ and $0<r\le 1$, then $B(0,r/4)\subseteq f(r\mathbb{D})$.*

We will deduce Theorem 4.1 from the following proposition, whose proof we will give after proving and deducing some consequences of Theorem 4.1.

Proposition 4.2. *If $f\in\mathcal{U}$, then $|a_2|\le 2$.*

Proof of Theorem 4.1. Suppose $f\colon\mathbb{D}\to D$ is a conformal transformation with $f\in\mathcal{U}$. Fix $z_0\notin D$. We will argue that $|z_0|\ge 1/4$, which will complete the proof of the theorem with $r=1$.

Let

$$\widetilde{f}(z)=\frac{z_0f(z)}{z_0-f(z)}.$$

As $f\in\mathcal{U}$, we have that

$$\widetilde{f}(0)=0 \quad \text{and} \quad \widetilde{f}'(0)=\frac{z_0^2f'(0)}{z_0^2}=1.$$

Also, $\widetilde{f}$ is a conformal transformation as it is given as a composition of conformal transformations. Therefore $\widetilde{f}\in\mathcal{U}$. If we write $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then we have that

$$\widetilde{f}(z)=z+\left(a_2+\frac{1}{z_0}\right)z^2+\cdots.$$

Consequently, Proposition 4.2 implies that

$$|a_2|\le 2 \quad \text{and} \quad \left|a_2+\frac{1}{z_0}\right|\le 2.$$

The triangle inequality thus implies that $|1/z_0| \le 4$ hence $|z_0| \ge 1/4$. This proves the theorem for $r = 1$. The theorem for general $r \in (0,1)$ can be deduced from the case that $r = 1$ by replacing f with the conformal transformation $z \mapsto f(rz)/r$. ☐

Corollary 4.3. *Suppose that $D, \widetilde{D}$ are domains in $\mathbb{C}$, $z \in D$, $\widetilde{z} \in \widetilde{D}$, and $f\colon D \to \widetilde{D}$ is a conformal transformation with $f(z) = \widetilde{z}$. Then*

$$\frac{\widetilde{d}}{4d} \le |f'(z)| \le \frac{4\widetilde{d}}{d}$$

where $d = \mathrm{dist}(z, \partial D)$ and $\widetilde{d} = \mathrm{dist}(\widetilde{z}, \partial \widetilde{D})$.

Proof. By translation, we may assume that $z = \widetilde{z} = 0$. Let

$$\widetilde{f}(w) = \frac{f(dw)}{df'(0)}.$$

Then $\widetilde{f} \in \mathcal{U}$. By Theorem 4.1, we have that $B(0, r/4) \subseteq \widetilde{f}(r\mathbb{D})$ for all $0 < r \le 1$. This implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $w \in \mathbb{D} \setminus (1-\delta)\mathbb{D}$ we have

$$\left| \frac{f(dw)}{df'(0)} \right| = |\widetilde{f}(w)| \ge \frac{1}{4} - \epsilon.$$

Therefore

$$\frac{\widetilde{d}}{d|f'(0)|} \ge \inf_{w \in \mathbb{D} \setminus (1-\delta)\mathbb{D}} |\widetilde{f}(w)| \ge \frac{1}{4} - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, by rearranging the above we see that

$$\frac{4\widetilde{d}}{d} \ge |f'(0)|.$$

This implies the upper bound. The lower bound follows from the same argument with f^{-1} in place of f. ☐

Proposition 4.4. *Suppose that $f \in \mathcal{U}$. Then*

$$\mathrm{area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

Proof. Fix $r \in (0,1)$ and let $\gamma(\theta) = f(re^{i\theta})$ for $\theta \in [0, 2\pi]$. Then we have that

$$\begin{aligned}
\frac{1}{2i} \int_\gamma \overline{z} dz &= \frac{1}{2i} \int_\gamma (x - iy)(dx + idy) \\
&= \frac{1}{2i} \int_\gamma (x - iy)dx + (ix + y)dy \\
&= \frac{1}{2i} \iint_{f(r\mathbb{D})} 2i dx dy \quad \text{(Green's formula)} \\
&= \mathrm{area}(f(r\mathbb{D})).
\end{aligned}$$

We also have that

$$\begin{aligned}\frac{1}{2i}\int_\gamma \overline{z}dz &= \frac{1}{2i}\int_0^{2\pi} \overline{f(re^{i\theta})}f'(re^{i\theta})ire^{i\theta}d\theta\\ &= \frac{1}{2i}\int_0^{2\pi}\left(\sum_{n=1}^\infty \overline{a_n}r^n e^{-i\theta n}\right)\left(\sum_{n=1}^\infty a_n n r^{n-1}e^{i\theta(n-1)}\right)ire^{i\theta}d\theta\\ &= \pi\sum_{n=1}^\infty r^{2n}|a_n|^2 n.\end{aligned}$$

Sending $r \to 1$ proves the result. $\square$

We will now complete the proof of Theorem 4.1 by checking that if $f(z) = z + \sum_{n=2}^\infty a_n z^n$ then $|a_2| \leq 2$. This is a special case of a famous conjecture in complex analysis called "Bieberbach's conjecture", which states that $|a_n| \leq n$ for all n, and was posed by Bieberbach in 1916. This conjecture was proved by de Branges in 1985. His proof makes use of the Loewner equation. In fact, the Loewner equation was considered by Loewner in order to prove the Bieberbach conjecture.

Definition 4.5. We say that a connected compact set $K \subseteq \mathbb{C}$ is a *compact hull* if $\mathbb{C}\setminus K$ is connected and K consists of more than a single point.

If K is a compact hull, then the Riemann mapping theorem implies that there exists a unique conformal transformation $F\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus K$ which fixes ∞ and has positive derivative at ∞. Equivalently, $\lim_{z\to\infty} F(z)/z > 0$.

Note that $F(z) = 1/f(1/z)$ where f is the unique conformal transformation $\mathbb{D} \to I(\mathbb{C}\setminus K)$ with $f(0)$ and $f'(0) > 0$ where here $I(z) = 1/z$ is the inversion map. Note also that

$$\frac{F(z)}{z} = \frac{1}{zf(1/z)} \to \frac{1}{f'(0)} > 0 \quad \text{as} \quad z \to \infty.$$

We let $\mathcal{H}$ be the set compact hulls containing 0 with $\lim_{z\to\infty} F(z)/z = 1$. If $K \in \mathcal{H}$, then the Laurent expansion of F takes the form

$$F(z) = z + b_0 + \sum_{n=1}^\infty \frac{b_n}{z^n}.$$

Proposition 4.6. *If $K \in \mathcal{H}$, then*

$$\text{area}(K) = \pi\left(1 - \sum_{n=1}^\infty n|b_n|^2\right).$$

As the right hand side must be non-negative, we in particular have that $\sum_{n=1}^\infty n|b_n|^2 \leq 1$.

Proof. Let $r > 1$ and let $K_r = F(r\overline{\mathbb{D}})$ and $\gamma(\theta) = F(re^{i\theta})$. Arguing as in the proof of Proposition 4.4, we have that

$$\text{area}(K_r) = \frac{1}{2i}\int_\gamma \overline{z}dz$$

$$= \frac{1}{2i}\int_0^{2\pi} \overline{F(re^{i\theta})}F'(re^{i\theta}) ire^{i\theta} d\theta$$
$$= \pi\left(r^2 - \sum_{n=1}^\infty n|b_n|^2 r^{-2n}\right).$$

Taking a limit as $r \to 1$, the left hand side converges to area(K) and the right hand side converges to $\pi(1-\sum_{n=1}^\infty n|b_n|^2)$, as desired. □

Lemma 4.7. *If $f \in \mathcal{U}$, there exists an odd function $h \in \mathcal{U}$ (i.e., $h(-z) = -h(z)$) such that $h(z)^2 = f(z^2)$.*

Proof. Let

$$\widetilde{f}(z) = \begin{cases} f(z)/z & \text{if} \quad z \neq 0 \\ f'(0) & \text{if} \quad z = 0. \end{cases}$$

Then $\widetilde{f}$ is non-zero and conformal in $\mathbb{D}$, which implies that there exists a function g with $g(z)^2 = \widetilde{f}(z)$. Let $h(z) = zg(z^2)$. Then h is odd with $h(z)^2 = f(z^2)$. Also, $h(0) = 0$ and $h'(0) = 1$. Note that if $h(z_1) = h(z_2)$, then $z_1 g(z_1^2) = z_2 g(z_2^2)$. By squaring both sides, we thus have that $z_1^2 g(z_1^2)^2 = z_2^2 g(z_2^2)^2$. Since $g(z_j^2)^2 = \widetilde{f}(z_j^2)$, this in turn implies that $z_1^2 \widetilde{f}(z_1^2) = z_2^2 \widetilde{f}(z_2^2)$. By the definition of $\widetilde{f}$, we have $f(z_1^2) = f(z_2^2)$ which implies that $z_1^2 = z_2^2$. Inserting this into the equation $z_1 g(z_1^2) = z_2 g(z_2^2)$ implies that $z_1 = z_2$. Therefore $h \in \mathcal{U}$. □

Proof of Proposition 4.2. Suppose that $f \in \mathcal{U}$ and that h is as in the previous lemma. As h is odd, it follows that its series expansion about 0 does not have any even powers of z. That is,

$$h(z) = z + c_3 z^3 + c_5 z^5 + \cdots.$$

Moreover, the identity $h(z)^2 = f(z^2)$ implies that

$$z^2 + a_2 z^4 + \cdots = (z + c_3 z^3 + c_5 z^5 + \cdots)^2 = z^2 + 2c_3 z^4 + \cdots.$$

In particular, $c_3 = a_2/2$. Let $g(z) = 1/h(1/z)$. Then we have that

$$g(z) = \frac{1}{z^{-1} + (a_2/2)z^{-3} + \cdots} = \frac{z}{1 + (a_2/2)z^{-2} + \cdots} = z\left(1 - \frac{a_2}{2}z^{-2} + \cdots\right) = z - \frac{a_2}{2}z^{-1} + \cdots.$$

Proposition 4.6 implies that $|a_2/2|^2 \leq 1$ which in turn implies that $|a_2| \leq 2$, as desired. □

5. Half-plane capacity

Definition 5.1. A set $A \subseteq \mathbb{H}$ is called a *compact* $\mathbb{H}$*-hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. We let $\mathcal{Q}$ be the collection of compact $\mathbb{H}$-hulls.

In this section, we will be interested in

- Analyzing the "correct" conformal transformation $g_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ and

- A notion of "size" for $A \in \mathcal{Q}$ (half-plane capacity).

Proposition 5.2. *For each $A \in \mathcal{Q}$, there exists a unique conformal transformation $g_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ with $|g_A(z) - z| \to 0$ as $z \to \infty$.*

In order to prove Proposition 5.2, we will need to make use of the so-called Schwarz reflection principle.

Proposition 5.3 (Schwarz reflection principle)**.** *Let $D \subseteq \mathbb{H}$ be a simply connected domain and let $\phi \colon D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Then ϕ extends by reflection to a conformal transformation on $D^* = D \cup \{\overline{z} : z \in D\} \cup \{x \in \partial\mathbb{H} : D \text{ is a neighborhood of } x \text{ in } \mathbb{H}\}$ by setting $\phi(\overline{z}) = \overline{\phi(z)}$.*

We will not provide a proof of Proposition 5.3.

Proof of Proposition 5.2. The Riemann mapping theorem implies that there exists a conformal transformation $g \colon \mathbb{H} \setminus A \to \mathbb{H}$. By post-composing $\mathbb{H}$ with a conformal transformation $\mathbb{H} \to \mathbb{H}$ if necessary, we may assume without loss of generality that $|g(z)| \to \infty$ as $|z| \to \infty$ (i.e., g fixes ∞). By Schwarz reflection, we can extend g to a conformal transformation defined on $\mathbb{C} \setminus (\{\overline{z} : z \in A\} \cup \overline{A})$ by setting $g(\overline{z}) = \overline{g(z)}$. By performing a series expansion for $1/g(1/z)$, we see that g admits the Laurent expansion

$$g(z) = b_{-1}z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

If $z \in \mathbb{R}$, then $\overline{z} = z$ and $g(z) = g(\overline{z}) = \overline{g(z)}$. That is, if $z \in \mathbb{R} \setminus \overline{A}$ then $g(z) \in \mathbb{R}$. Consequently,

$$b_{-1}z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} = \overline{b_{-1}}z + \overline{b_0} + \sum_{n=1}^{\infty} \frac{\overline{b_n}}{z^n} \quad \text{for all} \quad z \in \mathbb{R} \setminus \overline{A}.$$

This implies that $b_j = \overline{b_j}$ for each j. In other words, each b_j is real. Set

$$g_A(z) = \frac{g(z) - b_0}{b_{-1}}.$$

As $b_{-1}, b_0 \in \mathbb{R}$, we have that $g_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ is a conformal transformation with $|g_A(z) - z| \to 0$ as $z \to \infty$. This completes the proof of existence.

To see the uniqueness, suppose that $\widetilde{g}_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ is another conformal transformation such that $|\widetilde{g}_A(z) - z| \to 0$ as $z \to \infty$. Then $\widetilde{g}_A \circ g_A^{-1}$ is a conformal transformation $\mathbb{H} \to \mathbb{H}$. This implies that there exists $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ such that

$$\widetilde{g}_A \circ g_A^{-1}(z) = \frac{az + b}{cz + d}.$$

Since $|\widetilde{g}_A \circ g_A^{-1}(z) - z| \to 0$ as $z \to \infty$, it follows that $a = c = 1$ and $b = d = 0$. That is, $\widetilde{g}_A \circ g_A^{-1}(z) = z$ which implies that $\widetilde{g}_A = g_A$. $\square$

Definition 5.4. Suppose that $A \in \mathcal{Q}$. The *half-plane capacity* of A is defined by

$$\mathrm{hcap}(A) = \lim_{z \to \infty} z(g_A(z) - z).$$

Equivalently, we have that

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + \sum_{n=2}^{\infty} \frac{b_n}{z^n}.$$

One should think of $\mathrm{hcap}(A)$ as a notion of "size" for A. We will shortly show that it is non-negative and monotone.

Example 5.5. Recall that $z \mapsto \sqrt{z^2+4t}$ is a conformal transformation $\mathbb{H} \setminus [0, 2\sqrt{t}i] \to \mathbb{H}$ with $|\sqrt{z^2+4t} - z| \to 0$ as $z \to \infty$. Note that

$$\sqrt{z^2+4t} = z + \frac{2t}{z} + \cdots .$$

Therefore $\mathrm{hcap}([0, 2\sqrt{t}i]) = 2t$.

Example 5.6. The map $z \mapsto z + 1/z$ maps $\mathbb{H} \setminus \overline{\mathbb{D}} \to \mathbb{H}$ and $|(z+1/z) - z| \to 0$ as $z \to \infty$. Therefore $\mathrm{hcap}(\mathbb{H} \cap \overline{\mathbb{D}}) = 1$.

We are now going to collect several properties of the half-plane capacity.

(i) **Scaling.** Suppose that $r > 0$, $A \in \mathcal{Q}$. Then $\mathrm{hcap}(rA) = r^2 \mathrm{hcap}(A)$. To see this, we note that $g_{rA}(z) = r g_A(z/r)$. Indeed, $r g_A(z/r)$ is a conformal transformation $\mathbb{H} \setminus (rA) \to \mathbb{H}$ with $|r g_A(z/r) - z| \to 0$ as $z \to \infty$ since g_A has this property. Therefore that $r g_A(z/r) = g_{rA}$ follows from the uniqueness part of Proposition 5.2. The scaling property thus follows as

$$r g_A(z/r) = r\left(z/r + \frac{\mathrm{hcap}(A)}{z/r} + \cdots \right) = z + \frac{r^2 \mathrm{hcap}(A)}{z} + \cdots .$$

(ii) **Translation invariance.** Suppose that $x \in \mathbb{R}$ and $A \in \mathcal{Q}$. Then $\mathrm{hcap}(A+x) = \mathrm{hcap}(A)$. To see this, we note that $g_A(z-x) + x$ is a conformal transformation $\mathbb{H} \setminus (A+x) \to \mathbb{H}$ with $|g_A(z-x) + x - z| \to 0$ as $z \to \infty$. Translation invariance thus follows by arguing as in the proof of the scaling property.

(iii) **Monotonicity.** Suppose that $A, \widetilde{A} \in \mathcal{Q}$ with $A \subseteq \widetilde{A}$. Then we have that $g_{\widetilde{A}} = g_{g_A(\widetilde{A} \setminus A)} \circ g_A$ since $g_{g_A(\widetilde{A} \setminus A)}$ is a conformal transformation $\mathbb{H} \setminus g_A(\widetilde{A} \setminus A) \to \mathbb{H}$ which looks like the identity at ∞ and likewise for g_A. Therefore $g_{g_A(\widetilde{A} \setminus A)} \circ g_A$ is a conformal transformation $\mathbb{H} \setminus \widetilde{A} \to \mathbb{H}$ which looks like the identity at ∞. As

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + \cdots \quad \text{and} \quad g_{g_A(\widetilde{A} \setminus A)} = z + \frac{\mathrm{hcap}(g_A(\widetilde{A} \setminus A))}{z} + \cdots$$

it follows that

$$g_{\widetilde{A}}(z) = g_{g_A(\widetilde{A} \setminus A)} \circ g_A(z) = z + \frac{\mathrm{hcap}(A) + \mathrm{hcap}(g_A(\widetilde{A} \setminus A))}{z} + \cdots .$$

We conclude that

$$\mathrm{hcap}(\widetilde{A}) = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(\widetilde{A} \setminus A)).$$

Upon showing that hcap ≥ 0, this will imply that $\mathrm{hcap}(\widetilde{A}) \geq \mathrm{hcap}(A)$. That is, hcap is monotone.

By combining the scaling and monotonicity properties of the half-plane capacity, we note that if $A \in \mathcal{Q}$ and $A \subseteq r\overline{\mathbb{D}} \cap \mathbb{H}$, then we have that

$$\mathrm{hcap}(A) \leq \mathrm{hcap}(r\overline{\mathbb{D}} \cap \mathbb{H}) = r^2 \mathrm{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = r^2.$$

We now turn to derive a representation for the half-plane capacity in terms of Brownian motion, which in particular implies that the half-plane capacity is non-negative.

Proposition 5.7. *Suppose that $A \in \mathcal{Q}$, B is a complex Brownian motion, and $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$ is the first exit time of B from $\mathbb{H} \setminus A$.*

(i) *For all $z \in \mathbb{H} \setminus A$,* $\mathrm{Im}(z - g_A(z)) = \mathbb{E}_z[\mathrm{Im}(B_\tau)]$.
(ii) $\mathrm{hcap}(A) = \lim_{y \to \infty} y \mathbb{E}_{iy}[\mathrm{Im}(B_\tau)]$.
(iii) $\mathrm{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)] \sin(\theta) d\theta$.

Proof. Note that $\phi(z) = \mathrm{Im}(z - g_A(z))$ is harmonic in $\mathbb{H} \setminus A$ as it is the imaginary part of a complex differentiable function. As $g_A(z) = z + \mathrm{hcap}(A)/z + \cdots$ and $\mathrm{Im}(g_A(z)) = 0$ for $z \in \partial(\mathbb{H} \setminus A)$, it follows that ϕ is bounded and continuous. Therefore (i) follows from Theorem 3.1.

Note that

$$\begin{aligned} \mathrm{hcap}(A) &= \lim_{z \to \infty} z(g_A(z) - z) \\ &= \lim_{y \to \infty} iy(g_A(iy) - iy). \end{aligned}$$

The proof of Proposition 5.2 implies that $\mathrm{hcap}(A)$ is real (as the coefficients in the series expansion of g_A are real). Taking real parts of both sides, we thus see that

$$\mathrm{hcap}(A) = \lim_{y \to \infty} y \mathrm{Im}(iy - g_A(iy)).$$

Therefore (ii) follows from (i).

Part (iii) is on Example Sheet 1. □

Before we proceed to derive some estimates for g_A, we pause to discuss the conformal invariance of Brownian motion. Roughly, this says that if B is a complex Brownian motion and f is a conformal transformation, then the random process $f(B)$ is a Brownian motion up to a random time-change. This statement can be checked directly in the special case that $f(z) = cz + d$ for $c, d \in \mathbb{C}$ (i.e., f can be thought of as first performing a rotation, then a scaling, then a translation) because one can check directly from the definition of complex Brownian motion then it is rotationally invariant, scale invariant (up to a time change), and translation invariant. Conformal transformations locally behave like such f, which is why this fact is intuitive. We now give a formal statement:

Theorem 5.8. *Let $D, \widetilde{D}$ be domains and let $f\colon D \to \widetilde{D}$ be a conformal transformation. Let $B, \widetilde{B}$ be complex Brownian motions starting from $z \in D$, $\widetilde{z} = f(z) \in \widetilde{D}$, respectively. Let*

$$\tau = \inf\{t \geq 0 : B_t \notin D\} \quad \textit{and} \quad \widetilde{\tau} = \inf\{t \geq 0 : \widetilde{B}_t \notin \widetilde{D}\}$$

be the exit times of $B, \widetilde{B}$ from $D, \widetilde{D}$, respectively. Set

$$\tau' = \int_0^\tau |f'(B_s)|^2 ds \quad \textit{and} \quad \sigma(t) = \inf\left\{s \geq 0 : \int_0^s |f'(B_r)|^2 dr = t\right\} \quad \textit{for} \quad t < \tau'.$$

With $B_t' = f(B_{\sigma(t)})$, we have that

$$(\tau', B_t' : t < \tau') \stackrel{d}{=} (\widetilde{\tau} : \widetilde{B}_t : t < \widetilde{\tau}).$$

Theorem 5.8 will be given as a problem on an example sheet in Stochastic Calculus. It is proved by applying Itô's formula, the Cauchy-Riemann equations, and the Lévy characterization of Brownian motion.

We can use Theorem 5.8 to deduce the form of the exit distribution of a complex Brownian motion from a simply connected domain D. Since we will only be concerned with exit distributions, we emphasize that the random time-change in Theorem 5.8 will not play a role. Here are a few cases that will be important for what follows:

- If B is a complex Brownian motion in $\mathbb{D}$ starting from 0, then its first exit distribution is given by the uniform distribution on $\partial\mathbb{D}$. This follows because complex Brownian motion is rotationally invariant.
- Using Theorem 5.8 and applying a conformal transformation $\mathbb{D} \to \mathbb{D}$ which takes 0 to a given point $z \in \mathbb{D}$, one can show that the density (with respect to Lebesgue measure on $\partial\mathbb{D}$) of the first exit distribution of a complex Brownian motion starting from z at the point $e^{i\theta} \in \partial\mathbb{D}$ is given by
$$\frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2} \quad \text{for} \quad \theta \in [0, 2\pi).$$
This is on Example Sheet 1.
- Again using Theorem 5.8, one can see that the first exit distribution of a complex Brownian motion starting from $z = x + iy \in \mathbb{H}$ from $\mathbb{H}$ has density with respect to Lebesgue measure on $\mathbb{R}$ given by
$$\frac{1}{\pi}\frac{y}{(x-u)^2+y^2} \quad \text{for} \quad u \in \partial\mathbb{H}.$$
This is also on Example Sheet 1.

For $A \in \mathcal{Q}$, we let

$$\text{rad}(A) = \sup\{|z| : z \in A\}.$$

That is, $\text{rad}(A)$ is the diameter of the smallest ball centered at the origin which contains A.

Proposition 5.9. *Suppose that $A \in \mathcal{Q}$, B is a complex Brownian motion, and $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$. Then*

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\frac{1}{2} - \mathbb{P}_{iy}[B_\tau \in [x, \infty)] \right) \quad \text{if} \quad x > \mathrm{rad}(A) \quad \text{and}$$
$$g_A(x) = \lim_{y \to \infty} \pi y \left(\mathbb{P}_{iy}[B_\tau \in (-\infty, x]] - \frac{1}{2} \right) \quad \text{if} \quad x < -\mathrm{rad}(A).$$

Proof. We will first prove the result in the special case that $A = \emptyset$ and then deduce the result in the general case. If $A = \emptyset$, then we have that

$$\begin{aligned}
&\lim_{y \to \infty} \pi y \left(\frac{1}{2} - \mathbb{P}_{iy}[B_\tau \in [x, \infty)] \right) \\
&= \lim_{y \to \infty} \pi y \mathbb{P}_{iy}[B_\tau \in [0, x]] \\
&= \lim_{y \to \infty} \pi y \int_0^x \frac{y}{\pi(s^2 + y^2)} ds \quad \text{(by Example Sheet 1, Problem 2)} \\
&= x \quad \text{(by dominated convergence).}
\end{aligned}$$

This proves the result in the case that $A = \emptyset$ and $x \geq 0$. The case that $x \leq 0$ is analogous.

We now turn to prove the result in the case that $A \neq \emptyset$. We write $g_A = u_A + i v_A$. Let $\sigma = \inf\{t \geq 0 : B_t \notin \mathbb{H}\}$. By the conformal invariance of Brownian motion, we have that

$$\begin{aligned}
\mathbb{P}_{iy}[B_\tau \in [x, \infty)] &= \mathbb{P}_{g_A(iy)}[B_\sigma \in [g_A(x), \infty)] \\
&= \mathbb{P}_{i v_A(iy)}[B_\sigma \in [g_A(x) - u_A(iy), \infty)].
\end{aligned}$$

Since $g_A(z) - z \to 0$ as $z \to \infty$, it follows that we have both

$$\frac{v_A(iy)}{y} \to 1 \quad \text{and} \quad y u_A(iy) \to 0 \quad \text{as} \quad y \to \infty.$$

Consequently, it follows that

$$\left| \mathbb{P}_{i v_A(iy)}[B_\sigma \in [g_A(x) - u_A(iy), \infty)] - \mathbb{P}_{iy}[B_\sigma \in [g_A(x), \infty)] \right| = o(y^{-1}) \quad \text{as} \quad y \to \infty.$$

Combining everything proves the result for $x > \mathrm{rad}(A)$. The result for $x < -\mathrm{rad}(A)$ is analogous. □

Corollary 5.10. *Suppose that $A \in \mathcal{Q}$ with $\mathrm{rad}(A) \leq 1$. Then*

$$x \leq g_A(x) \leq x + \frac{1}{x} \quad \text{if} \quad x > 1$$
$$x + \frac{1}{x} \leq g_A(x) \leq x \quad \text{if} \quad x < -1.$$

Moreover, if $A \in \mathcal{Q}$ then $|g_A(z) - z| \leq 3\mathrm{rad}(A)$ for all $z \in \mathbb{H} \setminus A$.

Proof. This is Example Sheet 1, Problem 9. □

Proposition 5.11. *There exists $c > 0$ such that for all $A \in \mathcal{Q}$ and $|z| \geq 2\mathrm{rad}(A)$ we have that*
$$\left| g_A(z) - z - \frac{\mathrm{hcap}(A)}{z} \right| \leq c \frac{\mathrm{rad}(A)\mathrm{hcap}(A)}{|z|^2}.$$

Proof. By scaling, we may assume without loss of generality that $\mathrm{rad}(A) = 1$. Throughout, we let
$$h(z) = z + \frac{\mathrm{hcap}(A)}{z} - g_A(z).$$
Our goal is then to bound $|h(z)|$. We will proceed by bounding the modulus of the imaginary part of h and then deduce the bound for h itself using the Cauchy-Riemann equations. To this end, we let
$$v(z) = \mathrm{Im}(h(z)) = \mathrm{Im}(z - g_A(z)) - \frac{\mathrm{Im}(z)\mathrm{hcap}(A)}{|z|^2}.$$
Let B be a complex Brownian motion and let $\sigma = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus \overline{\mathbb{D}}\}$. We also let $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$. For $\theta \in [0, \pi]$, we let $p(z, e^{i\theta})$ be the density with respect to Lebesgue measure at $e^{i\theta}$ for B_σ. It follows from the strong Markov property for B at time σ together with part (i) of Proposition 5.7 that
$$\mathrm{Im}(z - g_A(z)) = \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)] p(z, e^{i\theta}) d\theta.$$
Recall that
$$p(z, e^{i\theta}) = \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} \sin(\theta) \left(1 + O(|z|^{-1})\right) \quad \text{(Example Sheet 1, Problem 3)} \tag{5.1}$$
$$\mathrm{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)] \sin(\theta) d\theta \quad \text{(part (iii) of Proposition 5.7)}. \tag{5.2}$$
We thus have that
$$\begin{aligned} |v(z)| &= \left| \mathrm{Im}(z - g_A(z)) - \frac{\mathrm{Im}(z)}{|z|^2} \mathrm{hcap}(A) \right| \\ &= \left| \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)] p(z, e^{i\theta}) d\theta - \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} \int_0^\pi \mathbb{E}_{e^{i\theta}} \mathrm{Im}(B_\tau) \sin(\theta) d\theta \right| \quad \text{(by (5.2))} \\ &\leq c \frac{\mathrm{hcap}(A)\mathrm{Im}(z)}{|z|^3} \quad \text{(by (5.1))}, \end{aligned}$$
where $c > 0$ is a constant.

As v is harmonic (as it is the imaginary part of a complex differentiable function), it follows from Example Sheet 1, Problem 8 that we have for a constant $c > 0$ both
$$|\partial_x v(z)| \leq c \frac{\mathrm{hcap}(A)}{|z|^3} \quad \text{and} \quad |\partial_y v(z)| \leq c \frac{\mathrm{hcap}(A)}{|z|^3}.$$
By the Cauchy-Riemann equations, this implies that (possibly increasing the value of c)
$$|h'(z)| \leq c \frac{\mathrm{hcap}(A)}{|z|^3}. \tag{5.3}$$

Hence,

$$\begin{aligned} |h(iy)| &= \left| \int_y^\infty h'(is) ds \right| \quad (\text{as } h(iy) \to 0 \text{ as } y \to \infty) \\ &\leq \int_y^\infty |h'(is)| ds \\ &\leq c \frac{\text{hcap}(A)}{y^2} \quad (\text{by } (5.3)), \end{aligned}$$

with another possible increase in the value of c in the last inequality. This proves the bound for $z = iy$. For general $z = re^{i\theta}$ with $r \geq 2\text{rad}(A)$, we can integrate along $\partial(r\mathbb{D})$ using the bound (5.3) to see that

$$|h(z)| \leq |h(ir)| + c\frac{\text{hcap}(A)}{r^2},$$

which completes the proof. □

6. The chordal Loewner equation

The purpose of this section is to derive the chordal Loewner ODE. We begin by stating the so-called Beurling estimate (without proof), which is very useful in practice for proving bounds for the behavior of a conformal map near the domain boundary.

Theorem 6.1 (Beurling estimate)**.** *There exists a constant $c > 0$ such that the following is true. Suppose that B is a complex Brownian motion and $A \subseteq \overline{\mathbb{D}}$ is connected with $0 \in A$ and $A \cap \partial\mathbb{D} \neq \emptyset$. Then*

$$\mathbb{P}_z[B([0,\tau]) \cap A = \emptyset] \leq c|z|^{1/2} \tag{6.1}$$

where $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{D}\}$.

The upper bound in Theorem 6.1 is attained when A is the line segment $[-i, 0]$. To see that this is the case, one that a conformal map which takes $\mathbb{D} \setminus [-i, 0]$ to $\mathbb{H}$ which fixes 0 behaves like the square root map $z \mapsto \sqrt{z}$ near 0 (up to a rotation). (Indeed, the square root map is a conformal transformation $\mathbb{C} \setminus [0, \infty) \to \mathbb{H}$.)

As mentioned above, we will not prove Theorem 6.1. We note that it is not difficult to obtain a weaker version of Theorem 6.1 with some exponent $\alpha > 0$ in place of the exponent $1/2$ which appears on the right side of (6.1). This follows because a complex Brownian motion starting from $-3ir/4$ in an annulus $A = B(0, r) \setminus \overline{B(0, r/2)}$ has a positive chance (uniformly in $r > 0$) of disconnecting 0 from ∞ before leaving A.

Proposition 6.2. *There exists a constant $c > 0$ so that the following is true. Suppose that $A, \widetilde{A} \in \mathcal{Q}$ with $A \subseteq \widetilde{A}$ and $\widetilde{A} \setminus A$ is connected. Then*

$$\text{diam}(g_A(\widetilde{A} \setminus A)) \leq c \begin{cases} d^{1/2} r^{1/2} & \textit{if} \quad d \leq r \\ \text{rad}(\widetilde{A}) & \textit{if} \quad d > r \end{cases}$$

where $d = \mathrm{diam}(\widetilde{A} \setminus A)$ *and* $r = \sup\{\mathrm{Im}(z) : z \in \widetilde{A}\}$.

Proof. By scaling, we may assume without loss of generality that $r = 1$. If $d \geq 1$, then the result follows since the last part of Corollary 5.10 implies that $|g_A(z) - z| \leq 3\mathrm{rad}(A)$ hence $\mathrm{diam}(g_A(\widetilde{A} \setminus A)) \leq \mathrm{diam}(\widetilde{A}) + 6\mathrm{rad}(A) \leq 8\mathrm{rad}(\widetilde{A})$.

Now suppose that $d < 1$. Let B be a complex Brownian motion starting from iy, $y \geq 2$, and let $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$. Let z be such that $U = B(z, d) \supseteq \widetilde{A} \setminus A$. For $B([0, \tau])$ to intersect U, it must:

(i) Hit $B(z, 1)$ before leaving $\mathbb{H} \setminus A$. By Example Sheet 1, Problem 2, this occurs with probability at most c/y where $c > 0$ is a constant.
(ii) Given that (i) happens, it must visit U before leaving $\mathbb{H} \setminus A$. By the Beurling estimate (Theorem 6.1), this occurs with probability at most $cd^{1/2}$ where $c > 0$ is a constant.

Combining, this implies that (for a possibly larger value of $c > 0$)

$$\limsup_{y \to \infty} y\mathbb{P}_{iy}[B([0,\tau]) \cap U \neq \emptyset] \leq cd^{1/2}.$$

Let $\sigma = \inf\{t \geq 0 : B_t \notin \mathbb{H}\}$. By the conformal invariance of Brownian motion (recall the end of the proof of Proposition 5.9), this implies that

$$\limsup_{y \to \infty} y\mathbb{P}_{iy}[B([0,\tau]) \cap g_A(\widetilde{A} \setminus A)] \leq cd^{1/2}.$$

Since $g_A(\widetilde{A} \setminus A)$ is connected, it follows from Example Sheet 1, Problem 11 that $\mathrm{diam}(g_A(\widetilde{A} \setminus A)) \leq cd^{1/2}$ for a constant $c > 0$. ☐

Suppose that $(A_t) = (A_t)_{t \geq 0}$ is a family of compact $\mathbb{H}$-hulls. We say that (A_t) is

(i) *non-decreasing* if $0 \leq s \leq t < \infty$ implies that $A_s \subseteq A_t$
(ii) *locally growing* if for every $T, \epsilon > 0$ there exists $\delta > 0$ such that $0 \leq s \leq t \leq s + \delta \leq T$ implies that $\mathrm{diam}(g_s(A_t \setminus A_s)) \leq \epsilon$
(iii) *parameterized by half-plane capacity* if $\mathrm{hcap}(A_t) = 2t$ for all $t \geq 0$.

Let $\mathcal{A}$ be the collection of families of compact $\mathbb{H}$-hulls which satisfy (i)–(iii).

For $T > 0$, we also let $\mathcal{A}_T$ be the collection of families of compact $\mathbb{H}$-hulls which satisfy (i)–(iii) but are only defined on the interval $[0, T]$ (so that $\mathcal{A} = \mathcal{A}_\infty$).

Example 6.3. Proposition 6.2 implies that if γ is a simple curve in $\mathbb{H}$ starting from 0, then $A_t = \gamma([0, t])$ is a family of compact $\mathbb{H}$-hulls which satisfy (i) and (ii) above. By Example Sheet 1, Problem 11, we can reparameterize γ (i.e., perform a time-change) so that (A_t) is parameterized by half-plane capacity. Upon performing this time change, we have that (A_t) is in $\mathcal{A}$.

Theorem 6.4. *Suppose that (A_t) is in $\mathcal{A}$ with $A_0 = \emptyset$. For each $t \geq 0$, let $g_t = g_{A_t}$. There exists $U\colon [0,\infty) \to \mathbb{R}$ continuous such that*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Proof. Note that $\cap_{s>t}\overline{g_t(A_s)}$ contains a single point since (A_t) is locally growing. Call this point U_t. It is not difficult to see that in fact U_t is continuous in t since (A_t) is locally growing.

Recall from Proposition 5.11 that if $A \in \mathcal{Q}$ then

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O\left(\frac{\mathrm{hcap}(A)\mathrm{rad}(A)}{|z|^2}\right). \tag{6.2}$$

If $x \in \mathbb{R}$, then as $g_{A+x}(z) - x = g_A(z-x)$, it follows from (6.2) that

$$g_A(z) = g_{A+x}(z+x) - x = z + \frac{\mathrm{hcap}(A)}{z+x} + \mathrm{hcap}(A)\mathrm{rad}(A+x)O\left(\frac{1}{|z+x|^2}\right). \tag{6.3}$$

Fix $\epsilon > 0$. Note that $\mathrm{hcap}(g_t(A_{t+\epsilon} \setminus A_t)) = 2\epsilon$. For $0 \leq s \leq t$, let $g_{s,t} = g_t \circ g_s^{-1}$. Applying (6.3) with $A = g_t(A_{t+\epsilon} \setminus A_t)$ and $x = -U_t$ and using that $\mathrm{rad}(g_t(A_{t+\epsilon} \setminus A_t) - U_t) \leq \mathrm{diam}(g_t(A_{t+\epsilon} \setminus A_t))$, we thus see that

$$g_{t,t+\epsilon}(z) = z + \frac{2\epsilon}{z - U_t} + 2\epsilon\mathrm{diam}(g_t(A_{t+\epsilon} \setminus A_t))O\left(\frac{1}{|z-U_t|^2}\right).$$

We thus have that

$$\begin{aligned} g_{t+\epsilon}(z) - g_t(z) &= (g_{t,t+\epsilon} - g_{t,t}) \circ g_t(z) \\ &= \frac{2\epsilon}{g_t(z) - U_t} + 2\epsilon\mathrm{diam}(g_t(A_{t+\epsilon} \setminus A_t))O\left(\frac{1}{|g_t(z) - U_t|^2}\right) \end{aligned}$$

Dividing both sides by ϵ, sending $\epsilon \to 0$, and using that (A_t) is locally growing, we thus see that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}$$

as desired. □

Theorem 6.4 implies that we can encode a family (A_t) in $\mathcal{A}$ with $A_0 = \emptyset$ in terms of a continuous, real-valued function U.

Conversely, if U is a continuous, real-valued function and we let

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

then A_t given by the complement in $\mathbb{H}$ of the domain of g_t is a family in $\mathcal{A}$ with $A_0 = \emptyset$. This is Example Sheet 1, Problem 12.

The function U is called the "Loewner driving function" for (A_t).

7. Derivation of the Schramm-Loewner evolution

The purpose of this section is to explain the derivation and definition of SLE.

Definition 7.1. Suppose that (A_t) is a random family in $\mathcal{A}$ encoded with the Loewner driving function U. We say that (A_t) satisfies the *conformal Markov property* if the following is true. For each $t \geq 0$, let $\mathcal{F}_t = \sigma(U_s : s \leq t)$. Then:

(i) The conditional law of $(g_t(A_{t+s}) - U_t)_{s \geq 0}$ given $\mathcal{F}_t$ is equal to that of $(A_s)_{s\geq 0}$. (Markov property)
(ii) For each $r > 0$, $(rA_{t/r^2}) \stackrel{d}{=} (A_t)$. (Scale invariance)

Note that (i) is equivalent to the statement that, given $\mathcal{F}_t$, $(U_{t+s} - U_t)_{s\geq 0}$ has the same distribution as $(U_s)_{s\geq 0}$. That is, U has stationary, independent increments. As U is continuous, this implies that there exists $\kappa \geq 0$ and $a \in \mathbb{R}$ such that $U_t = \sqrt{\kappa} B_t + at$ where B is a standard Brownian motion.

By (ii), we have for $r > 0$ that

$$rU_{t/r^2} = \sqrt{\kappa} r B_{t/r^2} + ra(t/r^2) = \sqrt{\kappa}\widetilde{B} + at/r \stackrel{d}{=} U_t$$

where $\widetilde{B}$ is a standard Brownian motion. The only way that this can be the case is if $a = 0$.

Combining, we have just obtained Schramm's theorem.

Theorem 7.2 (Schramm). *If (A_t) satisfies the conformal Markov property, then there exists $\kappa \geq 0$ such that $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion.*

For $\kappa > 0$, SLE_κ is the random family of hulls (A_t) which are obtained by solving the Loewner equation with $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion.

SLE_0 corresponds to the case $U_t \equiv 0$ for all $t \geq 0$, which corresponds to the curve $A_t = [0, 2\sqrt{t}i]$.

Remark 7.3. (i) It turns out that SLE_κ is *generated* by a continuous curve γ. That is, $\mathbb{H} \setminus A_t$ is equal to the unbounded component of $\mathbb{H} \setminus \gamma([0,t])$ for each $t \geq 0$. Equivalently, A_t is equal to the set obtained by "filling in" the holes cut off from ∞ by $\gamma|_{[0,t]}$. This result was first proved by Rohde-Schramm. In the rest of this course, we will take it as an assumption.
(ii) The behavior of SLE_κ depends strongly on κ. We will show later that SLE_κ is simple for $\kappa \in (0, 4]$, self-intersecting for $\kappa \in (4, 8)$, and space-filling for $\kappa \geq 8$.
(iii) As we proved just above, SLE_κ is singled out by the conformal Markov property. This is motivated from conjectures in the physics literature which regarding the behavior of scaling limits of discrete models in two dimensions (percolation, loop-erased random walk, etc...)
(iv) The main tool to analyze SLE_κ is stochastic calculus, which we will review next.

8. Stochastic calculus review

The general setting that we shall have in mind is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)$ which satisfies the *usual conditions*:

(i) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets
(ii) $(\mathcal{F}_t)$ is right-continuous, i.e., $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$.

The basic object in stochastic calculus is the *continuous semi-martingale*. This is a process X_t which can be written as a sum $M_t + A_t$ where M_t is a continuous local martingale and A_t is a process of bounded variation.

The following concepts from stochastic calculus will be important for this course:

- The stochastic integral
- The quadratic variation
- Itô's fomrula
- Lévy characterization of Brownian motion
- Stochastic differential equations

8.1. **The stochastic integral.**

The stochastic integral of a previsible process H_t against a semi-martingale $X_t = M_t + A_t$ is defined by setting

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dX_s.$$

The first integral on the right hand is an Itô integral and is a continuous local martingale. The second integral is a Lebesgue-Stieljes integral and is a process of bounded variation. The Itô integral is defined and constructed in a way which is similar to the Riemann integral. It exists due to the cancellation which arises since M_t is a continuous local martingale, even though M_t does not have finite variation.

8.2. **Quadratic variation.**

The quadratic variation of a continuous local martingale M is

$$[M]_t = \lim_{n\to\infty} \sum_{k=0}^{\lceil 2^n t\rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2.$$

It is the unique non-decreasing continuous process such that

$$M_t^2 - [M]_t$$

is a continuous local martingale. The quadratic variation of a continuous process of finite variation vanishes. So,

$$[X]_t = [M + A]_t = [M]_t.$$

Also,

$$[\int H_s dM_s]_t = \int_0^t H_s^2 d[M]_s.$$

8.3. **Itô's formula.** Itô's formula is the stochastic calculus analog of the fundamental theorem of calculus. To motiviate it, suppose that $f \in C^(\mathbb{R})$. If $t \geq 0$ and $0 = t_0 < \cdots < t_n = t$ is a partition of $[0, t]$, then we can write

$$\begin{aligned} f(t) &= f(0) + \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1}) \big) \\ &= f(0) + \sum_{k=1}^{n} \big(f'(t_{k-1})(t_k - t_{k-1}) + o(t_k - t_{k-1}) \big) \quad \text{(Taylor's theorem)} \\ &\to f(0) + \int_0^t f'(s) ds \quad \text{as} \quad \max_{1 \leq k \leq n} (t_k - t_{k-1}) \to 0. \end{aligned}$$

Now suppose that B is a standard Brownian motion with $B_0 = 0$. Then we can write

$$\begin{aligned} f(B_t) &= f(0) + \sum_{k=1}^{n} \big(f(B_{t_k}) - f(B_{t_{k-1}}) \big) \\ &= f(0) + \sum_{k=1}^{n} \Big(f'(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) + \frac{1}{2} f''(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2 + o((B_{t_k} - B_{t_{k-1}})^2) \Big) \\ &\to f(0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds \quad \text{as} \quad \max_{1 \leq k \leq n} (t_k - t_{k-1}) \to 0. \end{aligned}$$

We have derived a special case of Itô's formula:

$$f(B_t) = f(0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds.$$

Here is a more general version. Suppose that $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. The first variable is the time variable and the second variable is the spatial variable. If $X_t = M_t + A_t$ is a continuous semimartingale, then Itô's formula states that:

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_s f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) dX_s + \frac{1}{2} \int_0^t \partial_x^2 f(s, X_s) d[M]_s.$$

We can rewrite this as:

$$\begin{aligned} f(t, X_t) =& f(0, X_0) + \int_0^t \partial_x f(s, X_s) dM_s + \Big(\int_0^t \partial_s f(s, X_s) ds + \\ & \int_0^t \partial_x f(s, X_s) dA_s + \frac{1}{2} \int_0^t \partial_x^2 f(s, X_s) d[M]_s \Big). \end{aligned}$$

The first integral is the martingale part of the semimartinagle decomposition of $f(t, X_t)$ and the other integrals together are the bounded variation part.

8.4. **Lévy characterization.** Suppose that M is a continuous local martingale. The Lévy characterization of Brownian motion states that M is a Brownian motion if and only if $[M]_t = t$ for all $t \geq 0$. It is proved by using Itô's formula to show that the process $e^{i\theta M_t + \theta^2/2[M]_t}$ is a continuous martingale.

8.5. **Stochastic differential equations.** Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ together with $(\mathcal{F}_t)$ is a probability space satisfying the usual conditions. Let B be a standard Brownian motion which is adapted to $(\mathcal{F}_t)$. If b, σ are measurable functions, then we say that a continuous semimartingale X_t adapted to $(\mathcal{F}_t)$ satisfies the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

provided

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \quad \text{for all} \quad t \geq 0.$$

It will be proved in Stochastic Calculus that this SDE has a unique solution when b, σ are Lipschitz functions.

9. Phases of SLE

Suppose that $X = (B^1, \ldots, B^d)$ is a d-dimensional Brownian motion. In other words, $B^1, \ldots, B^d$ are independent standard Brownian motions. Let

$$Z_t = \|X_t\|^2 = (B_t^1)^2 + \cdots + (B_t^d)^2.$$

By Itô's formula, we have that

$$Z_t = (B_t^1)^2 + \cdots + (B_t^d)^2 = Z_0 + 2\int_0^t B_s^1 dB_s^1 + \cdots + 2\int_0^t B_s^d dB_s^d + dt.$$

Let

$$Y_t = \int_0^t \frac{1}{Z_s^{1/2}} B_s^1 dB_s^1 + \cdots + \int_0^t \frac{1}{Z_s^{1/2}} B_s^d dB_s^d.$$

Then Y_t is a continuous local martingale with

$$\begin{aligned}
[Y]_t &= \left[\int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^1 dB_s^1 + \cdots + \int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^d dB_s^d\right]_t \\
&= \left[\int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^1 dB_s^1\right]_t + \cdots + \left[\int_0^\cdot \frac{1}{Z_s^{1/2}} B_s^d dB_s^d\right]_t \\
&= \int_0^t \frac{1}{Z_s}(B_s^1)^2 ds + \cdots + \int_0^t \frac{1}{Z_s}(B_s^d)^2 ds \\
&= t.
\end{aligned}$$

Consequently, the Lévy characterization implies that $Y_t = \widetilde{B}_t$ where $\widetilde{B}$ is a standard Brownian motion. This allows us to write

$$Z_t = Z_0 + 2\int_0^t Z_s^{1/2} d\widetilde{B}_s + dt.$$

Equivalently,

$$dZ_t = 2Z_t^{1/2} d\widetilde{B}_t + d \cdot dt.$$

This it the "square Bessel SDE of dimension d" and we say that Z is a square Bessel process of dimension d. Sometimes, this is written as $Z_t \sim \mathrm{BESQ}^d$. This SDE has a solution for every $d \in \mathbb{R}$ which is defined at least up until the first time that the process hits 0. In particular, d need not be an integer.

By applying Itô's formula with $f(x) = \sqrt{x}$, we next see that

$$\begin{aligned} Z_t^{1/2} &= Z_0^{1/2} + \frac{1}{2}\int_0^t Z_s^{-1/2} dZ_s - \frac{1}{8}\int_0^t Z_s^{-3/2} d[Z]_s \\ &= Z_0^{1/2} + \widetilde{B}_t + \frac{d}{2}\int_0^t Z_s^{-1/2} ds - \frac{1}{2}\int_0^t Z_s^{-1/2} ds \\ &= Z_0^{1/2} + \left(\frac{d-1}{2}\right)\int_0^t Z_s^{-1/2} ds + \widetilde{B}_t. \end{aligned}$$

Thus $U_t = Z_t^{1/2}$ satisfies

$$U_t = U_0 + \left(\frac{d-1}{2}\right)\int_0^t \frac{1}{U_s} ds + \widetilde{B}_t.$$

Equivalently,

$$dU_t = \left(\frac{d-1}{2}\right)\frac{1}{U_t} dt + d\widetilde{B}_t.$$

This is the "Bessel SDE of dimension d" and we say that U is a Bessel process of dimension d. Sometimes this is written as $U_t \sim \mathrm{BES}^d$. As in the case of the square Bessel SDE, the Bessel SDE has a solution for every $d \in \mathbb{R}$ which is defined at least up until the first time that the process hits 0. So, as before, d need not be an integer.

Proposition 9.1. *Suppose that $d \in \mathbb{R}$ and $U_t \sim \mathrm{BES}^d$.*

(i) If $d < 2$, then U_t hits 0 a.s.
(ii) If $d \geq 2$, then U_t does not hit 0 a.s.

Proof. We will prove the proposition by considering the process U_t^{2-d}. By Itô's formula, we have that

$$\begin{aligned} U_t^{2-d} &= U_0^{2-d} + \int_0^t (2-d) U_s^{1-d} dU_s + \frac{1}{2}\int_0^t (2-d)(1-d) U_s^{-d} d[U]_s \\ &= U_0^{2-d} + \int_0^t (2-d) U_s^{1-d} d\widetilde{B}_s + \int_0^t \frac{(d-2)(d-1)}{2U_s} U_s^{1-d} ds + \frac{1}{2}\int_0^t (2-d)(1-d) U_s^{-d} ds \\ &= U_0^{2-d} + \int_0^t (2-d) U_s^{1-d} d\widetilde{B}_s. \end{aligned}$$

This proves that U_t^{2-d} is a continuous, local martingale. For each $a \in \mathbb{R}$, we let $\tau_a = \inf\{t \geq 0 : U_t = a\}$. If $0 \leq a < U_0 < b < \infty$, then the process $U_{t\wedge\tau_a\wedge\tau_b}^{2-d}$ is a bounded, continuous martingale. The optional stopping theorem thus implies that

$$U_0^{2-d} = \mathbb{E}[U_{\tau_a\wedge\tau_b}^{2-d}] = a^{2-d}\mathbb{P}[\tau_a < \tau_b] + b^{2-d}\mathbb{P}[\tau_b < \tau_a].$$

If $d < 2$, then we can take $a = 0$ to see that

$$U_0^{2-d} = b^{2-d}\mathbb{P}[\tau_b < \tau_0].$$

That is,

$$\mathbb{P}[\tau_b < \tau_0] = \left(\frac{U_0}{b}\right)^{2-d}.$$

By sending $b \to \infty$, we see that $\mathbb{P}[\tau_0 < \infty] = 1$. If $d > 2$, then we can write

$$\mathbb{P}[\tau_a < \tau_b] = \left(\frac{U_0}{a}\right)^{2-d} - \left(\frac{b}{a}\right)^{2-d}\mathbb{P}[\tau_b < \tau_a].$$

Taking a limit as $a \to 0$, we see that $\mathbb{P}[\tau_0 < \tau_b] = 0$ for any b. Therefore $\mathbb{P}[\tau_0 < \infty] = 0$. The case $d = 2$ is proved similarly but with $\log U_t$ in place of U_t^{2-d}. □

Suppose that (g_t) solves the chordal Loewner equation driven by $U_t = \sqrt{\kappa}B_t$ where B is a standard Brownian motion. That is,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Let γ be the curve which corresponds to the family of hulls encoded by (g_t). For each $x \in \mathbb{R}$, let $V_t^x = g_t(x) - U_t$ and let $\tau_x = \inf\{t \geq 0 : V_t^x = 0\}$. Then τ_x is the first time that x is cut off from ∞ by γ. Note that

$$dV_t^x = \frac{2}{g_t(x) - U_t}dt - dU_t = \frac{2}{V_t^x}dt - \sqrt{\kappa}dB_t.$$

Equivalently,

$$d(V_t^x/\sqrt{\kappa}) = \frac{2/\kappa}{V_t^x/\sqrt{\kappa}}dt + d\widetilde{B}_t \quad \text{where} \quad \widetilde{B}_t = -B_t.$$

That is, $V_t^x/\sqrt{\kappa}$ is a BES^d with

$$\frac{d-1}{2} = \frac{2}{\kappa}$$

hence

$$d = 1 + \frac{4}{\kappa}.$$

Note that $d \geq 2$ if and only if $\kappa \leq 4$. Consequently, $\tau_x < \infty$ if and only if $\kappa > 4$.

Proposition 9.2. SLE_κ *corresponds to a simple curve for* $\kappa \leq 4$. *It is self-intersecting for* $\kappa > 4$.

Proof. The above considerations imply that SLE_κ intersects $\partial\mathbb{H}$ if and only if $\kappa > 4$. Suppose that $t > 0$ is fixed. Then $s \mapsto g_t(\gamma(s+t)) - U_t$ is an SLE_κ curve. The proposition follows as, for each $t \geq 0$, intersection points between $\gamma|_{[t,\infty)}$ and $\gamma|_{[0,t]}$ correspond to points where the curve $s \mapsto g_t(\gamma(s+t)) - U_t$ hits the boundary. □

We are now going to show that SLE_κ for $\kappa \in (4, 8)$ cuts off regions from ∞ and that SLE_κ for $\kappa \geq 8$ fills the boundary and does not cut off regions from ∞. It will be shown on Example Sheet 2 that SLE_κ for $\kappa \geq 8$ in fact fills all $\mathbb{H}$ (i.e., is space-filling).

For the rest of this section, we will assume that $\kappa > 4$.

To this end, for $0 < x < y$, we let $g(x,y) = \mathbb{P}[\tau_x = \tau_y]$ be the probability that both x and y are cut off from ∞ at the same time. We make two observations about $g(x,y)$:

- $g(x,y) = g(1, y/x)$ since SLE_κ is scale-invariant.
- $g(1,r) \to 0$ as $r \to \infty$ since $\mathbb{P}[\tau_1 < t] \to 1$ as $t \to \infty$ and $\mathbb{P}[\tau_r < t] \to 0$ as $r \to \infty$ with t fixed.

We say that events A, B are *equivalent* if $\mathbb{P}[A \setminus B] = \mathbb{P}[B \setminus A] = 0$, i.e., A, B differ by an event of probability 0.

Lemma 9.3. *Fix $r > 1$. The event $\{\tau_1 = \tau_r\}$ is equivalent to the event*

$$E = \left\{ \sup_{t < \tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty \right\}.$$

Proof. Indeed, if E occurs then we cannot have that $\tau_1 < \tau_r$. Therefore $E \subseteq \{\tau_1 = \tau_r\}$. On the other hand, if $M > 0$, then we have that

$$\mathbb{P}\left[\tau_1 = \tau_r \,|\, \sup_{t < \tau_1} \frac{V_t^r - V_t^1}{V_t^1} \geq M \right] = \mathbb{P}[\tau_1 = \tau_r \,|\, \sigma_M < \tau_1]$$

where $\sigma_M = \inf\{t \geq 0 : (V_t^r - V_t^1)/V_t^1 \geq M\}$. By the scale-invariance of SLE_κ and the strong Markov property applied at the stopping time σ_M, we therefore have that

$$\mathbb{P}[\tau_1 = \tau_r \,|\, \sigma_M < \tau_1] = g(1, 1+M) \to 0 \quad \text{as} \quad M \to \infty.$$

This implies that

$$\mathbb{P}\left[\tau_1 = \tau_r, E^c\right] = 0,$$

which concludes the proof that $\{\tau_1 = \tau_r\}$ and E are equivalent. $\square$

Our goal now is to show that

$$\mathbb{P}[\sup_{t < \tau_1} (V_t^r - V_t^1)/V_t^1 < \infty]$$

is positive if $\kappa \in (4,8)$ and is equal to 0 if $\kappa \geq 8$. Let

$$Z_t = \log\left(\frac{V_t^r - V_t^1}{V_t^1} \right).$$

With $d = 1 + 4/\kappa$, we have by Itô's formula that

$$dZ_t = \left(\left(\frac{3}{2} - d \right) \frac{1}{(V_t^1)^2} + \left(\frac{d-1}{2} \right) \left(\frac{V_t^r - V_t^1}{(V_t^1)^2 V_t^r} \right) \right) dt - \frac{1}{V_t^1} dB_t \quad \text{with} \quad Z_0 = \log(r-1).$$

We are now going to perform a time-change to turn the local martingale part of Z_t into a standard Brownian motion. Let

$$\sigma(t) = \inf\left\{ u \geq 0 : \int_0^u \frac{1}{(V_s^1)^2} ds = t \right\}.$$

Then we have that

$$t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} ds \quad \text{hence} \quad dt = \frac{d\sigma(t)}{(V_{\sigma(t)}^1)^2}.$$

Note that the process

$$\widetilde{B}_t = -\int_0^{\sigma(t)} \frac{1}{V_s^1} dB_s$$

is a continuous local martingale with

$$[\widetilde{B}]_t = \left[-\int_0^{\sigma(\cdot)} \frac{1}{V_s^1} dB_s \right]_t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} ds = t.$$

Therefore the Lévy characterization implies that $\widetilde{B}$ is a standard Brownian motion. Thus, with $\widetilde{Z}_t = Z_{\sigma(t)}$, we have that

$$d\widetilde{Z}_t = \left(\left(\frac{3}{2} - d \right) + \left(\frac{d-1}{2} \right) \left(\frac{V_{\sigma(t)}^r - V_{\sigma(t)}^1}{V_{\sigma(t)}^r} \right) \right) dt + d\widetilde{B}_t.$$

Consequently,

$$\begin{aligned} \widetilde{Z}_t &= \widetilde{Z}_0 + \widetilde{B}_t + \left(\frac{3}{2} - d \right) t + \frac{d-1}{2} \int_0^t \frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} ds \\ &\geq \widetilde{Z}_0 + \widetilde{B}_t + \left(\frac{3}{2} - d \right) t. \end{aligned}$$

If $\kappa \geq 8$ then $d = 1 + 4/\kappa \leq 3/2$, in which case we have that

$$\widetilde{Z}_t \geq \widetilde{Z}_0 + \widetilde{B}_t.$$

Hence

$$\sup_{t \geq 0} \widetilde{Z}_t \geq \widetilde{Z}_0 + \sup_{t \geq 0} \widetilde{B}_t = \infty.$$

As $\sigma(\infty) = \tau_1$, we thus have that

$$\sup_{t < \tau_1} e^{Z_t} = \infty.$$

We conclude that $g(x, y) = 0$ for all $0 < x < y$ if $\kappa \geq 8$. We have just established the following.

Proposition 9.4. *An* SLE_κ *for* $\kappa \geq 8$ *almost surely fills* $\partial\mathbb{H}$. *In particular, such a process does not cut regions off from* ∞.

Now suppose that $\kappa \in (4, 8)$. Fix $\epsilon > 0$ and assume that $r = 1 + \epsilon/2$. Note $\widetilde{Z}_0 = \log(r-1) = \log(\epsilon/2)$. Let

$$\tau = \inf\{t \geq 0 : \widetilde{Z}_t = \log \epsilon\}.$$

Then

$$\begin{aligned} \widetilde{Z}_{t\wedge\tau} &= \widetilde{Z}_0 + \widetilde{B}_{t\wedge\tau} + \left(\frac{3}{2} - d \right) t \wedge \tau + \left(\frac{d-1}{2} \right) \int_0^{t\wedge\tau} \frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} ds \\ &\leq \widetilde{Z}_0 + \widetilde{B}_{t\wedge\tau} + \left(\frac{3}{2} - d \right) t \wedge \tau + \left(\frac{d-1}{2} \right) \int_0^{t\wedge\tau} e^{\widetilde{Z}_s} ds \end{aligned}$$

$$\leq \widetilde{Z}_0 + \widetilde{B}_{t\wedge\tau} \left(\left(\frac{3}{2} - d \right) + \left(\frac{d-1}{2} \right) \epsilon \right) t \wedge \tau$$
$$= \widetilde{Z}_0 + \widetilde{B}_{t\wedge\tau} + at \wedge \tau \quad \text{where} \quad a = \left(\frac{3}{2} - d \right) + \left(\frac{d-1}{2} \right) \epsilon.$$

Assume that $\epsilon > 0$ is taken to be sufficiently small so that $a < 0$ (recall that $d > 3/2$ since $\kappa \in (4,8)$). Let

$$Z_t^* = \widetilde{Z}_0 + \widetilde{B}_t + at.$$

Then

$$Z_{t\wedge\tau}^* \geq \widetilde{Z}_{t\wedge\tau}.$$

As Z_t^* is a Brownian motion with negative drift starting from $\log(\epsilon/2)$, it follows that

$$\mathbb{P}[\sup_{t\geq 0} Z_t^* < \log \epsilon] > 0.$$

Therefore

$$\mathbb{P}[\sup_{t\geq 0} \widetilde{Z}_t < \log \epsilon] > 0.$$

Hence

$$\mathbb{P}\left[\sup_{t<\tau_1} e^{Z_t} < \epsilon \right] > 0.$$

This implies that $g(1, 1+\epsilon/2) > 0$. It follows from the scale-invariance and Markov property for SLE_κ that then $g(x,y) > 0$ for all $0 < x < y$ as desired (see Example Sheet 2). We have just established the following:

Proposition 9.5. *An* SLE_κ *for* $\kappa \in (4,8)$ *almost surely cuts off regions from* ∞.

10. Locality of SLE_6

So far, we have only defined SLE_κ in $\mathbb{H}$ from 0 to ∞. If $D \subseteq \mathbb{C}$ is a simply connected domain and $x, y \in \partial D$ are distinct, then there exists a conformal transformation $\phi \colon \mathbb{H} \to D$ with $\phi(0) = x$ and $\phi(\infty) = y$. An SLE_κ γ in D from x to y is defined by taking it to be $\phi(\widetilde{\gamma})$ where $\widetilde{\gamma}$ is an SLE_κ in $\mathbb{H}$ from 0 to ∞. (It will be shown on Example Sheet 2 that this definition is well-defined.)

We will now analyze the question of which SLE_κ should correspond to the scaling limit of percolation.

Suppose that $D \subseteq \mathbb{C}$ is simply connected, $x, y \in \partial D$ are distinct. Consider $p = 1/2$ (critical) percolation on the hexagonal lattice with hexagons of size ϵ which intersect $\overline{D}$. We take the hexagons which intersect the clockwise (resp. counterclockwise) segment of ∂D from x to y to be all black (resp. white). With this choice of boundary conditions, there exists a unique interface γ^ϵ which connects x to y with black (resp. white) hexagons on its left (resp. right) side. (See Figure 10.1 for an illustration and Figure 1.2 for actual simulations in the special case of a lozenge shaped domain.)

It was conjectured (now proved by Smirnov) that the limit γ of γ^ϵ is *conformally invariant*. This means that if $\widetilde{D}$ is another simply connected domain, $\widetilde{x}, \widetilde{y} \in \partial\widetilde{D}$ are distinct, and $\psi \colon D \to \widetilde{D}$ is a

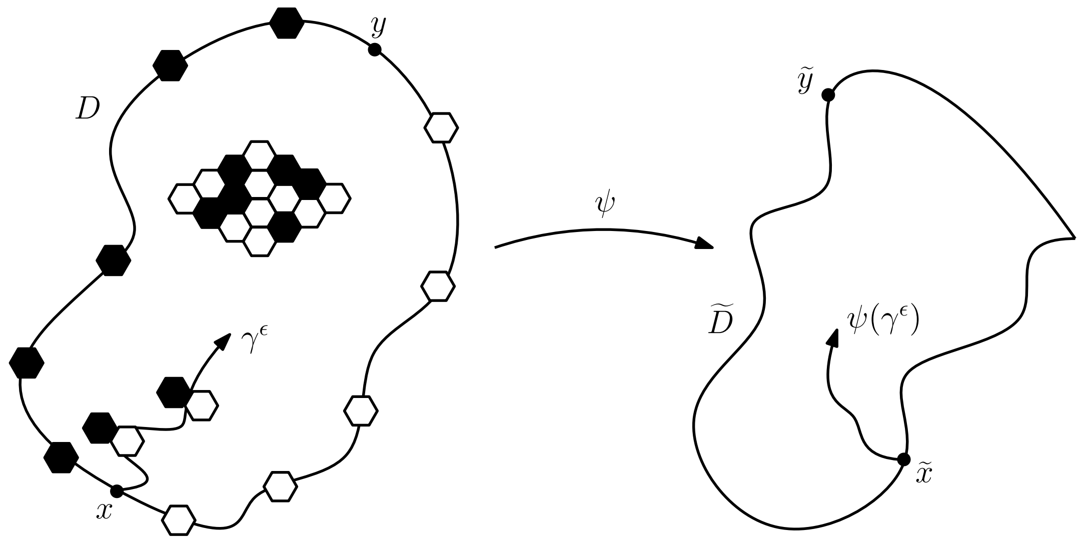

FIGURE 10.1. Percolation exploration γ^ϵ in the hexagonal lattice with hexagons of size ϵ in a simply connected domain D from x to y with black (resp. white) hexagons on the clockwise (resp. counterclockwise) arc of ∂D from x to y. Some representative hexagons are shown together with their colors. The scaling limit γ of γ^ϵ as $\epsilon \to 0$ was conjectured (now proved by Smirnov) to be conformally invariant, which means that if $\psi\colon D \to \widetilde{D}$ is a conformal transformation with $\widetilde{x} = \psi(x)$ and $\widetilde{y} = \psi(y)$, then the law of $\psi(\gamma) = \lim_{\epsilon\to 0} \psi(\gamma^\epsilon)$ is equal in distribution to the scaling limit of the percolation exploration in $\widetilde{D}$ from $\widetilde{x}$ to $\widetilde{y}$ with the corresponding black/white boundary conditions.

conformal transformation, then $\psi(\gamma)$ is equal in distribution to the scaling limit of percolation on $\widetilde{D}$ from $\widetilde{x}$ to $\widetilde{y}$ with boundary conditions analogous to those described just above. (See Figure 10.1 for an illustration.)

Also, percolation satisfies a natural Markov property (this is its spatial Markov property). Namely, if you condition on γ^ϵ up to a time t, then the conditional law of the remainder of the percolation interface is that of a percolation exploration in the remaining domain from $\gamma^\epsilon(t)$ to y. The reason for this is that in order to observe γ^ϵ, one need only observe the black (resp. white) hexagons which are on its left (resp. right) side.

If the scaling limit γ of the percolation exploration exists and it is conformally invariant, then the above considerations imply that it must satisfy the conformal Markov property. Therefore there must exist $\kappa \geq 0$ such that γ is an SLE_κ. We will now show that the only κ value which can correspond to the scaling limit of percolation is $\kappa = 6$.

Percolation possesses the extra property which is referred to as "locality". In special situation that we consider the percolation exploration on $\mathbb{H}$, it can be formulated as follows (but is indeed a very general principle). Suppose that D is a simply connected domain in $\mathbb{H}$ with 0 on its boundary. Then a percolation exploration in D with black (resp. white) boundary conditions on $\mathbb{R}_- \cap \partial D$ (resp. $\mathbb{R}_+ \cap \partial D$), run up until hitting $\partial D \setminus \partial\mathbb{H}$, has the same distribution as a percolation exploration in

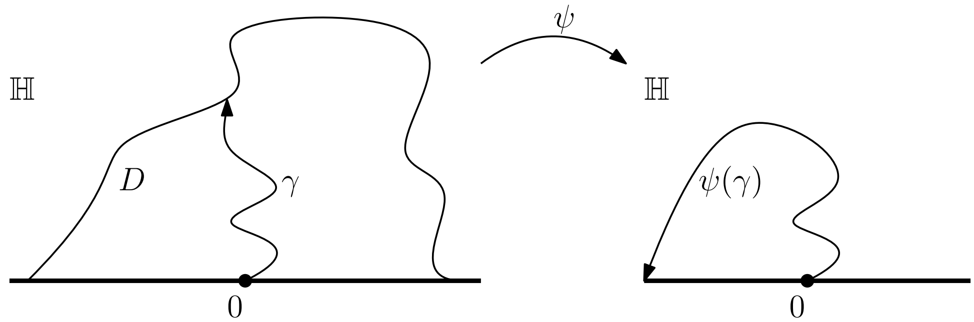

FIGURE 10.2. Illustration of the locality property for SLE_κ. Shown on the left is an SLE_κ curve γ in $\mathbb{H}$ from 0 to ∞ stopped upon leaving a simply connected domain $D \subseteq \mathbb{H}$ with $0 \in \partial D$. SLE_κ is said to satisfy the locality property if γ has the same distribution as an SLE_κ in D, stopping upon hitting $\partial D \setminus \partial \mathbb{H}$. Equivalently, if ψ is a conformal transformation $D \to \mathbb{H}$ fixing 0, then $\psi(\gamma)$ has the law of an SLE_κ in $\mathbb{H}$ from 0 to ∞, stopped upon hitting $\psi(\partial D \setminus \partial \mathbb{H})$. It turns out that locality holds if and only if $\kappa = 6$, which implies that the only SLE_κ which can correspond to the scaling limit of percolation is SLE_6.

all of $\mathbb{H}$ with black (resp. white) boundary conditions on $\mathbb{R}_-$ (resp. $\mathbb{R}_+$), also stopped upon hitting $\partial D \setminus \partial \mathbb{H}$.

Therefore, the corresponding SLE_κ should satisfy an analogous property. That is, we want to figure out for which value of κ the following is true. Suppose that $D \subseteq \mathbb{H}$ is a simply connected domain with 0 on its boundary. Let γ be an SLE_κ in $\mathbb{H}$ from 0 and consider γ stopped upon hitting $\partial D \setminus \partial \mathbb{H}$. Then we want that γ has the same law as an SLE_κ in D starting from 0 stopped at the analogous time. Equivalently, if $\psi \colon D \to \mathbb{H}$ is a conformal transformation with $\psi(0) = 0$, then we want that $\psi(\gamma)$ is an SLE_κ in $\mathbb{H}$. This is the so-called "locality property."

We will now show that locality holds if and only if $\kappa = 6$.

In order to establish this, we need to understand how the Loewner evolution changes when we apply a conformal transformation. Suppose that (A_t) is a non-decreasing family of compact $\mathbb{H}$-hulls which are locally growing and are parameterized by capacity and assume that $T > 0$ is such that $A_T \subseteq D$. For each $t \in [0,T]$, let $\widetilde{A}_t = \psi(A_t)$. Then $(\widetilde{A})_{t \in [0,T]}$ is a family of compact $\mathbb{H}$-hulls which are non-decreasing, locally growing, and with $\widetilde{A}_0 = \emptyset$.

For each $t \geq 0$, let $\widetilde{g}_t = g_{\widetilde{A}_t}$ be the unique conformal transformation $\mathbb{H} \setminus \widetilde{A}_t \to \mathbb{H}$ with $\widetilde{g}_t(z) - z \to 0$ as $z \to \infty$. Let $\widetilde{a}(t) = \mathrm{hcap}(\widetilde{A}_t)$. It will be shown on Example Sheet 2 that $(\widetilde{g}_t)$ satisfies

$$\partial_t \widetilde{g}_t(z) = \frac{\partial_t \widetilde{a}(t)}{\widetilde{g}_t(z) - \widetilde{U}_t}, \quad \widetilde{g}_0(z) = z \tag{10.1}$$

where $\widetilde{U}_t = \psi_t(U_t)$ for $\psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1}$, (g_t) the Loewner evolution associated with (A_t), and U_t its Loewner driving function. Also, (see Example Sheet 2)

$$\widetilde{a}(t) = \int_0^t 2(\psi_s'(U_s))^2 ds. \tag{10.2}$$

(The formula (10.2) is intuitive — and indeed derived — if one recalls the scaling property for half-plane capacity deduced earlier.)

We will want to apply Itô's formula to deduce the semi-martingale form of $\widetilde{U}_t = \psi_t(U_t)$. In order to do so, we need to identify the time-derivative of ψ_t evaluated at U_t.

Proposition 10.1. *The maps* (ψ_t) *satisfy*

$$\partial_t \psi_t(z) = 2\left(\frac{(\psi_t'(U_t))^2}{\psi_t(z) - \psi_t(U_t)} - \psi_t'(z)\frac{1}{z - U_t}\right).$$

Moreover, at $z = U_t$, *we have*

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

Proof. We have that

$$\begin{aligned}\partial_t \psi_t(z) &= (\partial_t \widetilde{g}_t)(\psi(g_t^{-1}(z))) + \widetilde{g}_t'(\psi(g_t^{-1}(z)))\psi'(g_t^{-1}(z))\partial_t(g_t^{-1}(z)) \\ &= \frac{2(\psi_t'(U_t))^2}{\psi_t(z) - \psi_t(U_t)} - \psi_t'(z)\frac{2}{z - U_t}.\end{aligned}$$

This proves the first assertion of the proposition, where we have used the identity

$$0 = \partial_t(g_t^{-1}(g_t(z))) = (\partial_t g_t^{-1})(g_t(z)) + (g_t^{-1})'(g_t(z))\frac{2}{g_t(z) - U_t}$$

in order to derive the formula for $\partial_t g_t^{-1}(z)$.

The second assertion of the proposition is on Example Sheet 2. □

Suppose that $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion. By Itô's formula, we have that

$$\begin{aligned}dU_t &= d\psi_t(U_t) \\ &= \left(\partial_t \psi_t(U_t) + \frac{\kappa}{2}\psi_t''(U_t)\right)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t \\ &= \left(-3\psi_t''(U_t) + \frac{\kappa}{2}\psi_t''(U_t)\right)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t \quad \text{(by Proposition 10.1)} \\ &= \frac{\kappa - 6}{2}\psi_t''(U_t)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t.\end{aligned}$$

We now let

$$\sigma(t) = \inf\{u \geq 0 : \int_0^u (\psi_s'(U_s))^2 ds = t\}.$$

Then

$$\partial_t \widetilde{g}_{\sigma(t)}(z) = \frac{2}{\widetilde{g}_{\sigma(t)} - \widetilde{U}_{\sigma(t)}}dt, \quad \widetilde{g}_{\sigma(0)}(z) = z.$$

Also, if we let $\widetilde{U}_t^* = \widetilde{U}_{\sigma(t)}$, then we have that

$$d\widetilde{U}_t^* = \frac{\kappa - 6}{2}\frac{\psi_{\sigma(t)}''(U_{\sigma(t)})}{(\psi_{\sigma(t)}'(U_{\sigma(t)}))^2}dt + \sqrt{\kappa}d\widetilde{B}_t$$

where

$$\widetilde{B}_t = \int_0^{\sigma(t)} \psi_s'(U_s) dB_s$$

is a standard Brownian motion (by the Lévy characterization). In particular, if $\kappa = 6$ then we have that $\widetilde{U}_t^* = \sqrt{6}\widetilde{B}_t$. That is, $(\widetilde{A}_{\sigma(t)})$ is equal in distribution to the family of hulls associated with an SLE_6. We have now obtained the following theorem:

Theorem 10.2. *If γ is an SLE_6 curve, then $\psi(\gamma)$ is an SLE_6 (up until first hitting $\psi(\partial D \setminus \partial \mathbb{H})$ and considered modulo a time-change).*

We conclude that SLE_6 is the only possible SLE curve which could describe the scaling limit of percolation.

11. The restriction property of $\mathrm{SLE}_{8/3}$

Our goal now is to show that $\mathrm{SLE}_{8/3}$ is the only SLE_κ which can arise as the scaling limit of a self-avoiding walk (SAW).

11.1. **The self-avoiding walk.** Suppose that $G = (V, E)$ is a graph with bounded maximal degree, $x \in V$, and $n \in \mathbb{N}$. The SAW in G starting from x of length n is the uniform measure on simple paths in G which start from x and have length n. Note that there are only a finite number of such paths (as the maximal degree is bounded) and under this probability measure we assign each such path equal weight.

The SAW was introduced in 1953 by Paul Flory, a Nobel prize winning chemist. The SAW on $\mathbb{Z}^d$ for $d \geq 5$ was shown by Hara and Slade to converge to a Brownian motion in the scaling limit. The same is conjectured to be true for the SAW on $\mathbb{Z}^4$. For $\mathbb{Z}^3$, it is not known which continuous object should describe the large scale behavior of the SAW. The rest of this section is focused on deriving the conjecture that the SAW on $\mathbb{Z}^2$ should converge to $\mathrm{SLE}_{8/3}$.

As we will explain below, the special property of the SAW which will single out the value $\kappa = 8/3$ is its so-called *restriction* property. This says that if $G' = (V', E')$ is a subgraph of G with $x \in V'$, then the SAW on G conditioned to stay in G' has the law of a SAW in G'. This just follows because the uniform measure restricts to the uniform measure on a smaller subset. To derive the $\mathrm{SLE}_{8/3}$/SAW conjecture, we will show that $\mathrm{SLE}_{8/3}$ is the only SLE_κ which satisfies a continuum version of the restriction property.

11.2. **Statement and characterization of the restriction property.** In the rest of what follows, we will assume that $\kappa \leq 4$ so that SLE_κ is simple. We also let $\mathcal{Q}_+$ consist of those $A \in \mathcal{Q}$ such that $\overline{A} \cap (-\infty, 0] = \emptyset$ and similarly let $\mathcal{Q}_-$ consist of those $A \in \mathcal{Q}$ such that $\overline{A} \cap [0, \infty) = \emptyset$. For $A \in \mathcal{Q}_\pm = \mathcal{Q}_+ \cup \mathcal{Q}_-$, we let $\psi_A = g_A - g_A(0)$. Then $\psi_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ is the unique conformal transformation with $\psi_A(0) = 0$ and $\psi_A(z)/z \to 1$ as $z \to \infty$.

We shall take as an assumption that SLE_κ is "transient". This means that if γ is an SLE_κ in $\mathbb{H}$ from 0 to ∞, then $\lim_{t\to\infty}\gamma(t)=\infty$ a.s. Since SLE_κ for $\kappa\le 4$ is simple, this implies that

$$0<\mathbb{P}[\gamma([0,\infty))\cap A=\emptyset]<1.$$

Let $V_A=\{\gamma([0,\infty))\cap A=\emptyset\}$. We say that an SLE_κ γ satisfies *restriction* if the conditional law of γ given V_A is that of an SLE_κ in $\mathbb{H}\setminus A$ from 0 to ∞. Equivalently, the conditional law of $\psi_A\circ\gamma$ given V_A is that of an SLE_κ in $\mathbb{H}$ from 0 to ∞.

We note that since γ is simple, its law (modulo time-change) is determined by the family of probabilities $A\mapsto\mathbb{P}[V_A]$ for $A\in\mathcal{Q}_\pm$.

Lemma 11.1. *Suppose there exists $\alpha>0$ so that $\mathbb{P}[V_A]=(\psi_A'(0))^\alpha$ for all $A\in\mathcal{Q}_\pm$. Then* SLE_κ *satisfies restriction.*

We will check that the criterion of Lemma 11.1 holds for $\mathrm{SLE}_{8/3}$ later in this section.

Proof of Lemma 11.1. Assume that $\mathbb{P}[V_A]=(\psi_A'(0))^\alpha$ for all $A\in\mathcal{Q}_\pm$. Suppose that $A,B\in\mathcal{Q}_\pm$. Then

$$\begin{aligned}\mathbb{P}[\psi_A\circ\gamma([0,\infty))\cap B=\emptyset\,|\,V_A]&=\frac{\mathbb{P}[\psi_A\circ\gamma([0,\infty))\cap B=\emptyset,\ \gamma([0,\infty))\cap A=\emptyset]}{\mathbb{P}[\gamma([0,\infty)\cap A=\emptyset]}\\&=\frac{\mathbb{P}[\gamma([0,\infty)\cap(A\cup\psi_A^{-1}(B))=\emptyset]}{\mathbb{P}[\gamma([0,\infty))\cap A=\emptyset]}\\&=\frac{(\psi_B'(0))^\alpha(\psi_A'(0))^\alpha}{(\psi_A'(0))^\alpha}\\&=(\psi_B'(0))^\alpha\\&=\mathbb{P}[V_B].\end{aligned}$$

This proves the result because the law of γ is characterized by the family of probabilities $B\mapsto\mathbb{P}[V_B]$ for $B\in\mathcal{Q}_\pm$. ☐

11.3. **Brownian excursions.** Suppose that $D\subseteq\mathbb{C}$ is a simply connected domain and $x,y\in\partial D$ are distinct. Informally, a *Brownian excursion* in D from x to y is a Brownian motion starting from x and conditioned to stay in D until exiting at y. Here we are conditioning on a zero probability event; we will make this precise just below using a limiting argument.

We will first define the Brownian excursion in $\mathbb{H}$ from 0 to ∞ and then extend the definition to other domains using conformal mapping and the conformal invariance of Brownian motion. The construction proceeds as follows:

- We suppose that $B=(B^1,B^2)$ is a complex Brownian motion starting from $i\epsilon$.
- We then condition B on the positive probability event that $\mathrm{Im}(B_t)=B_t^2$ hits $R>0$ large before hitting 0 (i.e., before B exits $\mathbb{H}$). Note that this event has probability ϵ/R.

- We then send $R \to \infty$ and then $\epsilon \to 0$.
- The limit $\widehat{B} = (\widehat{B}^1, \widehat{B}^2)$ is given by taking $\widehat{B}^1$ to be a standard Brownian motion in $\mathbb{H}$ and $\widehat{B}^2$ to be an independent BES^3 process starting from 0. (See Example Sheet 2.)

Proposition 11.2. *Suppose that $A \in \mathcal{Q}$ and g_A is as usual. If $z \in \mathbb{R} \setminus \overline{A}$, then we have that*

$$\mathbb{P}_z[\widehat{B}[0,\infty) \cap A = \emptyset] = g_A'(z).$$

Proof. For each $R > 0$, let $\mathcal{I}_R = \{z \in \mathbb{H} : \mathrm{Im}(z) = R\}$. Recall from Corollary 5.10 that $|g_A(z) - z| \leq 3\mathrm{rad}(A)$ for all $z \in \mathbb{H} \setminus A$. It follows that

$$g_A(\mathcal{I}_R) \subseteq \{z \in \mathbb{H} : R - 3\mathrm{rad}(A) \leq \mathrm{Im}(z) \leq R + 3\mathrm{rad}(A)\} \quad \text{for all} \quad R \geq 3\mathrm{rad}(A). \tag{11.1}$$

Let B be a complex Brownian motion and $\widehat{B}$ be a Brownian excursion. Let

$$\sigma_R = \inf\{t \geq 0 : \mathrm{Im}(B_t) = R\} \quad \text{and} \quad \widehat{\sigma}_R = \inf\{t \geq 0 : \mathrm{Im}(\widehat{B}_t) = R\}.$$

For $z \in \mathbb{H} \setminus A$, we note that

$$\begin{aligned} \mathbb{P}_z[\widehat{B}[0,\infty) \cap A = \emptyset] &= \lim_{R \to \infty} \mathbb{P}_z[\widehat{B}[0,\widehat{\sigma}_R] \cap A = \emptyset] \\ &= \lim_{R \to \infty} \frac{\mathbb{P}_z[B[0,\sigma_R] \cap (A \cup \mathbb{R}) = \emptyset]}{\mathbb{P}_z[B[0,\sigma_R] \cap \mathbb{R} = \emptyset]}. \end{aligned} \tag{11.2}$$

Note that the denominator is equal to $\mathrm{Im}(z)/R$ by the Gambler's ruin formula for Brownian motion. By the conformal invariance of Brownian motion and (11.1), the numerator satisfies the bound

$$\mathbb{P}_{g_A(z)}[B[0,\sigma_{R+3\mathrm{rad}(A)}] \cap \mathbb{R} = \emptyset] \leq \mathbb{P}_z[B[0,\sigma_R] \cap (A \cup \mathbb{R}) = \emptyset] \leq \mathbb{P}_{g_A(z)}[B[0,\sigma_{R-3\mathrm{rad}(A)}] \cap \mathbb{R} = \emptyset].$$

Applying the Gambler's ruin formula to the left and right sides of the inequality above, we thus see that

$$\frac{\mathrm{Im}(g_A(z)}{R + 3\mathrm{rad}(A)} \leq \mathbb{P}_z[B[0,\sigma_R] \cap (A \cup \mathbb{R}) = \emptyset] \leq \frac{\mathrm{Im}(g_A(z))}{R - 3\mathrm{rad}(A)}.$$

Inserting this formula into (11.2), we thus see that

$$\mathbb{P}_z[\widehat{B}[0,\infty) \cap A = \emptyset] = \frac{\mathrm{Im}(g_A(z))}{\mathrm{Im}(z)}.$$

The proposition follows by taking a limit as $\mathrm{Im}(z) \to 0$. $\square$

11.4. **Restriction Theorem for** $\mathrm{SLE}_{8/3}$**.** Recall from earlier that $V_A = \{\gamma[0,\infty) \cap A = \emptyset\}$ where $A \in \mathcal{Q}_\pm$ and $\gamma \sim \mathrm{SLE}_\kappa$ in $\mathbb{H}$ from 0 to ∞. Let $\mathcal{F}_t = \sigma(U_s : s \leq t)$ be the filtration of the driving process $U_t = \sqrt{\kappa} B_t$.

Consider the process $\widetilde{M}_t = \mathbb{P}[V_A \,|\, \mathcal{F}_t]$. Then $\widetilde{M}_t$ is a bounded martingale (as it is a conditional probability given an increasing family of σ-algebras) with $\widetilde{M}_0 = \mathbb{P}[V_A]$ and

$$\widetilde{M}_t \to \mathbf{1}_{V_A} \quad \text{as} \quad t \to \infty \quad \text{a.s.}$$

by the martingale convergence theorem. Let $\tau = \inf\{t \geq 0 : \gamma(t) \in A\}$. Then we have that

$$\begin{aligned}\widetilde{M}_t &= \mathbb{P}[V_A \,|\, \mathcal{F}_t] \\ &= \mathbb{P}[V_A \,|\, \mathcal{F}_t]\big(\mathbf{1}_{\{t<\tau\}} + \mathbf{1}_{\{t\geq\tau\}}\big) \\ &= \mathbb{P}[V_A \,|\, \mathcal{F}_t]\mathbf{1}_{\{t<\tau\}} \\ &= \mathbb{P}[V_{g_t(A)-U_t}]\mathbf{1}_{\{t<\tau\}} \quad \text{(by the conformal Markov property).}\end{aligned}$$

Observe that if M_t is another bounded $\mathcal{F}_t$-martingale with $M_t \to \mathbf{1}_{V_A}$ as $t \to \infty$ a.s., then $M_t = \widetilde{M}_t$ for all $t \geq 0$ a.s. Our goal then is to find a bounded martingale M_t with $M_t \to \mathbf{1}_{V_A}$ as $t \to \infty$. In view of Lemma 11.1, it is natural to guess that $\mathbb{P}[V_{g_t(A)-g_t(0)}]$ should be given by a power of a derivative of a conformal map. With this in mind, we will consider the process

$$M_t = (\psi'_{g_t(A)-U_t}(0))^{\alpha}\mathbf{1}_{\{t<\tau\}}$$

and then aim to show that the parameter α can be chosen appropriately so that it is a martingale for $\kappa = 8/3$ and it has the correct final value. Here, we recall that for $B \in \mathcal{Q}_{\pm}$, ψ_B is the unique conformal transformation $\mathbb{H} \setminus B \to \mathbb{H}$ with $\psi_B(0) = 0$ and $\psi_B(z)/z \to 1$ as $z \to \infty$. Writing $\psi_t = \widetilde{g}_t \circ \psi_A \circ g_t^{-1}$ where $\widetilde{g}_t = g_{\psi_A(\gamma(0,t]))}$ and (g_t) the Loewner flow associated with γ, we can write M_t as

$$M_t = (\psi'_t(U_t))^{\alpha}\mathbf{1}_{\{t<\tau\}}.$$

We have that (see Example Sheet 2),

$$\partial_t \psi'_t(U_t) = \frac{(\psi''_t(U_t))^2}{2\psi'_t(U_t)} - \frac{4}{3}\psi'''_t(U_t).$$

Thus by Itô's formula, we have that

$$dM_t = \alpha M_t\left(\frac{(\alpha-1)\kappa+1}{2}\frac{(\psi''_t(U_t))^2}{(\psi'_t(U_t))^2} + \left(\frac{\kappa}{2}-\frac{4}{3}\right)\frac{\psi'''_t(U_t)}{\psi'_t(U_t)}\right)dt + \alpha M_t\frac{\psi''_t(U_t)}{\psi'_t(U_t)}\sqrt{\kappa}dB_t.$$

If $\kappa = 8/3$ and $\alpha = 5/8$, then the above becomes

$$\frac{dM_t}{\alpha M_t} = \frac{\psi''_t(U_t)}{\psi'_t(U_t)}\sqrt{\kappa}dB_t.$$

Therefore M_t is a continuous local martingale. It is in fact a continuous martingale as Proposition 11.2 implies that it takes values in $[0,1]$. We will show that $M_t \to \mathbf{1}_{V_A}$ as $t \to \infty$ a.s. momentarily. Upon doing so, we will have obtained the following:

Theorem 11.3. $\mathrm{SLE}_{8/3}$ *satisfies the restriction property. Moreover, if* $\gamma \sim \mathrm{SLE}_{8/3}$, *then we have that*

$$\mathbb{P}[\gamma[0,\infty) \cap A = \emptyset] = (g'_A(0))^{5/8} \quad \textit{for all} \quad A \in \mathcal{Q}_{\pm}.$$

Before we proceed to the proof of Theorem 11.3, we first record the following interesting remark.

Remark 11.4. Suppose that $\gamma_1, \ldots, \gamma_8$ are independent $\mathrm{SLE}_{8/3}$ processes in $\mathbb{H}$ from 0 to ∞. Then Theorem 11.3 implies that

$$\mathbb{P}[\gamma_j[0,\infty) \cap A = \emptyset \ \forall 1 \leq j \leq 8] = (g_A'(0))^5 \quad \text{for all} \quad A \in \mathcal{Q}_\pm.$$

Also, if $\widehat{B}^1, \ldots, \widehat{B}^5$ are independent Brownian excursions in $\mathbb{H}$ from 0 to ∞, then

$$\mathbb{P}[\widehat{B}^j[0,\infty) \cap A = \emptyset \ \forall 1 \leq j \leq 5] = (g_A'(0))^5 \quad \text{for all} \quad A \in \mathcal{Q}_\pm.$$

This implies that the hull of $\gamma_1, \ldots, \gamma_8$ is equal in distribution to the hull of $\widehat{B}^1, \ldots, \widehat{B}^5$. Here, for a relatively closed subset $X \subseteq \mathbb{H}$, by the hull of X we mean X together with all of the bounded components of $\mathbb{H} \setminus X$.

Proof of Theorem 11.3. We will prove the result for $A \in \mathcal{Q}_+$. It suffices to consider the special case that A is bounded by a simple smooth curve $\beta\colon (0,1) \to \mathbb{H}$ (see Example Sheet 2). That is, $\mathbb{H} \cap \partial A = \beta(0,1)$. Let

$$M_t = (\psi_t'(U_t))^\alpha \mathbf{1}_{\{t<\tau\}}$$

where $\tau = \inf\{t \geq 0 : \gamma([0,t]) \cap A = \emptyset\}$. We emphasize that $\psi_t'(U_t)$ is the probability that a Brownian excursion in $\mathbb{H} \setminus \gamma([0,t])$ from $\gamma(t)$ to ∞ does not intersect A. In particular, this implies that $0 \leq M_t \leq 1$ for all $0 \leq t \leq \tau$. We note that $M_{t\wedge\tau}$ is a martingale. Therefore the martingale convergence theorem implies that

$$M_{t\wedge\tau} \to M_\infty \quad \text{as} \quad t \to \infty \quad \text{a.s.}$$

In particular, $0 \leq M_\infty \leq 1$.

Our goal is to show that $M_\infty = \mathbf{1}_{V_A}$. We will accomplish this in two steps:

Step 1: $M_{t\wedge\tau} \to 1$ on V_A a.s. as $t \to \infty$.
Step 2: $M_{t\wedge\tau} \to 0$ on V_A^c a.s. as $t \to \infty$.

By scaling, we may assume that $\sup\{\mathrm{Im}(w) : w \in A\} = 1$. For each $r > 2$, we let $\sigma_r = \inf\{t \geq 0 : \mathrm{Im}(\gamma(t)) = r\}$. We note that $\sigma_r < \infty$ a.s. for all $r > 0$ since $\mathrm{SLE}_{8/3}$ is transient.

Proof of Step 1: $M_{t\wedge\tau} \to 1$ on V_A as $t \to \infty$.

See Figure 11.1 for an illustration of the setup of the proof of Step 1.

Let $\widehat{B}$ be a Brownian excursion in $\mathbb{H} \setminus \gamma([0,\sigma_r])$ from $\gamma(\sigma_r)$ to ∞. Note that the limit as $\epsilon \to 0$ of the probability that $\widehat{B}$ hits A is given by $1 - \psi_{\sigma_r}'(U_{\sigma_r})$. This, in turn, is equal to

$$\lim_{\epsilon\to 0} \lim_{R\to\infty} \frac{\mathbb{P}_z[B[0,\tau_R] \subseteq \mathbb{H} \setminus \gamma[0,\sigma_r],\ B[0,\tau_R] \cap A \neq \emptyset]}{\mathbb{P}_z[B[0,\tau_R] \subseteq \mathbb{H} \setminus \gamma[0,\sigma_r]]} \tag{11.3}$$

where B is a complex Brownian motion, $z = \gamma(\sigma_r) + i\epsilon$ and

$$\tau_R = \inf\{t \geq 0 : \mathrm{Im}(B_t) = R\}.$$

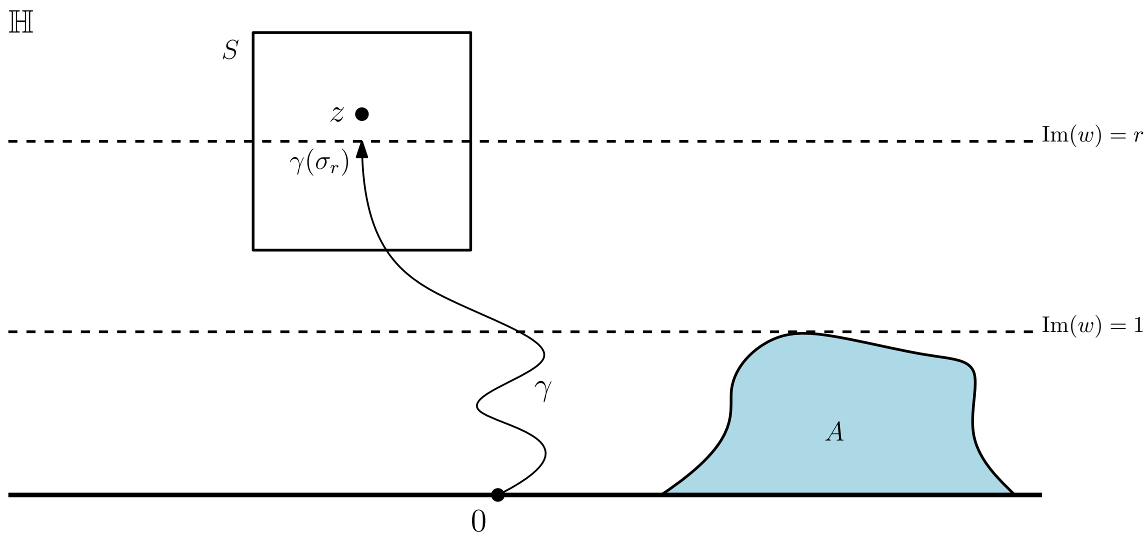

FIGURE 11.1. Illustration of the proof of Step 1 of Theorem 11.3.

Our goal is to show that the limit at most $Cr^{-1/2}$ for a constant $C > 0$. This will complete the proof of Step 1 as then we will have shown that

$$1 - Cr^{-1/2} \leq \psi'_{\sigma_r}(U_{\sigma_r}) \leq 1.$$

This, in turn, implies that

$$M_{\sigma_r \wedge \tau} \to 1 \quad \text{as} \quad r \to \infty \quad \text{on} \quad V_A$$

hence

$$M_{t \wedge \tau} \to 1 \quad \text{as} \quad t \to \infty \quad \text{on} \quad V_A$$

(as we know that $M_{t\wedge\tau}$ has a limit as $t \to \infty$).

Let $S = [-1,1]^2 + \gamma(\sigma_r)$ be the square of side-length 2 centered at $\gamma(\sigma_r)$ and let ℓ be the top of S. Let η be the first time that B leaves S. We note that the probability that $B(\eta) \in \ell$ is exactly equal to $1/4$ if B starts from $\gamma(\sigma_r)$ by symmetry. Consequently, $\mathbb{P}_z[B(\eta) \in \ell] > 1/4$. We similarly we have that $\mathbb{P}_w[B(\eta) \in \ell] < 1/4$ for all $w \in S$ with $\mathrm{Im}(w) \leq r$. Therefore

$$\begin{aligned}
1/4 &< \mathbb{P}_z[B(\eta) \in \ell] \\
&= \mathbb{P}_z[B(\eta) \in \ell,\ B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset] + \mathbb{P}_z[B(\eta) \in \ell,\ B[0,\eta] \cap \gamma[0,\sigma_r] \neq \emptyset] \\
&= \mathbb{P}_z[B(\eta) \in \ell \,|\, B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset]\mathbb{P}_z[B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset] + \\
&\quad \mathbb{P}_z[B(\eta) \in \ell \,|\, B[0,\eta] \cap \gamma[0,\sigma_r] \neq \emptyset]\mathbb{P}_z[B[0,\eta] \cap \gamma[0,\sigma_r] \neq \emptyset].
\end{aligned}$$

By the strong Markov property for B applied at the first time that it hits $\gamma[0,\sigma_r]$, it follows that

$$\mathbb{P}_z[B(\eta) \in \ell \,|\, B[0,\eta] \cap \gamma[0,\sigma_r] \neq \emptyset] < 1/4.$$

Therefore

$$\mathbb{P}_z[B(\eta) \in \ell \,|\, B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset] > 1/4.$$

Combining, we have that the denominator in (11.3) is at least

$$\frac{1}{4} \times \mathbb{P}[B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset] \times \frac{1}{R-r}. \tag{11.4}$$

Also, the strong Markov property and the Beurling estimate (Theorem 6.1) together imply that the probability that B starting from z hits A without hitting $\mathbb{R} \cup \gamma[0,\sigma_r]$ is at most $Cr^{-1/2}\mathbb{P}[B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset]$, for a constant $C > 0$. The probability that it subsequently hits the line $\{z \in \mathbb{H} : \mathrm{Im}(z) = R\}$ before leaving $\mathbb{H}$ is at most $1/R$. That is, the numerator in (11.3) is at most

$$Cr^{-1/2} \times \mathbb{P}[B[0,\eta] \cap \gamma[0,\sigma_r] = \emptyset] \times \frac{1}{R}. \tag{11.5}$$

The bounds (11.4) and (11.5) together imply that the limit in (11.3) is at most $Cr^{-1/2}$. As explained above, this completes the proof of Step 1.

Proof of Step 2: $M_{t\wedge\tau} \to 0$ on V_A^c as $t \to \infty$.

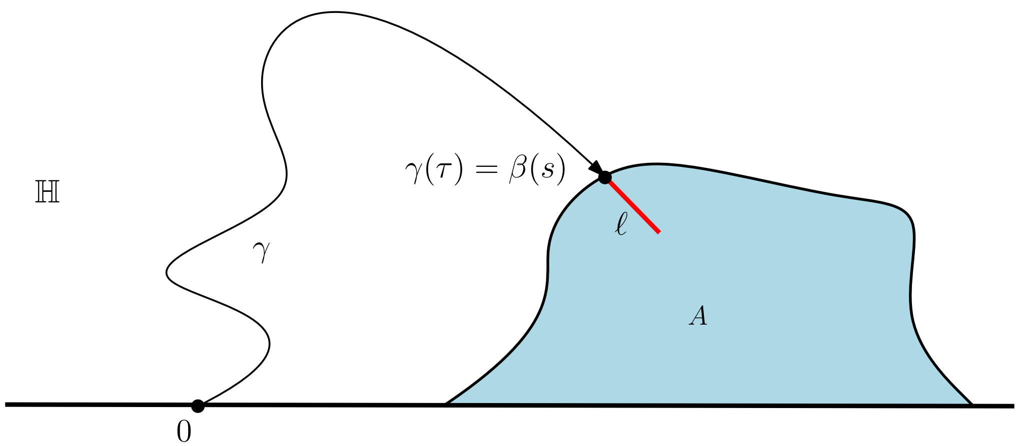

FIGURE 11.2. Illustration of the proof of Step 2 of Theorem 11.3.

See Figure 11.2 for an illustration of the proof of Step 2.

We note that there exists $s \in (0,1)$ such that $\gamma(\tau) = \beta(s)$. For each $M \in \mathbb{N}$, we let

$$\tau_M = \inf\{t \geq 0 : |\gamma(t) - \beta(s)| = 1/M\}.$$

Since β is a smooth curve, there exists $\delta > 0$ so that

$$\ell = [\beta(s), \beta(s) + \delta\vec{n}]$$

is contained in A where $\vec{n}$ is the inward pointing unit normal (i.e., pointing into A).

Note that there exists $p_0 > 0$ such that a Brownian motion starting from any point on ℓ has probability at least p_0 of hitting $\mathbb{R} \cup \gamma([0,\tau_M])$ for the first time on the right side of $\gamma([0,\tau_M])$. The

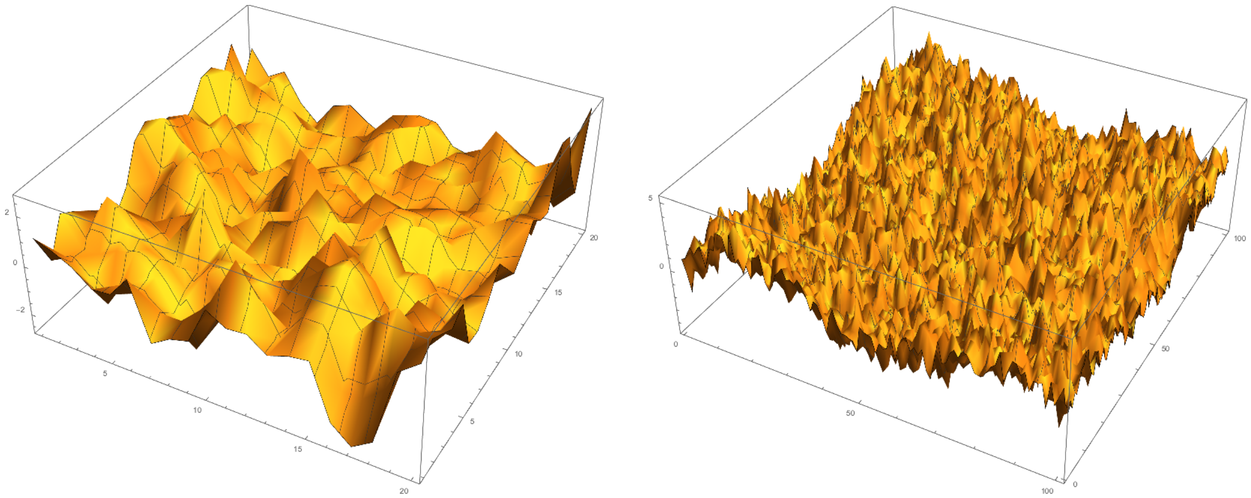

FIGURE 12.1. Discrete approximations of the Gaussian free field (GFF) on a 20×20 (left) and 100×100 (right) box in $\mathbb{Z}^2$.

same is also true for the left hand side of $\gamma([0, \tau_M])$. By the conformal invariance of Brownian motion, this implies that there exists $a > 0$ so that (see Example Sheet 2)

$$L_{\tau_M} = g_{\tau_M}(\ell) - U_{\tau_M} \subseteq \{w : \mathrm{Im}(w) \geq a|\mathrm{Re}(w)|\}. \tag{11.6}$$

That is, L_{τ_M} is contained in a sector in $\mathbb{H}$. From this, it is not difficult to see that the probability that a Brownian excursion starting from $\gamma(\tau_M)$ hits A tends to 1 as $M \to \infty$ (see Example Sheet 2). $\square$

12. The Gaussian free field

12.1. **Setup.** Throughout this section, we will make use of the following notation:

- C^∞ is the space of functions on $\mathbb{C}$ which are infinitely differentiable.
- C_0^∞ consists of those $f \in C_0^\infty$ with compact support.
- For a domain $D \subseteq \mathbb{C}$, $C_0^\infty(D)$ consists of those C_0^∞ functions with support in D.

Suppose that $f, g \in C_0^\infty$. The *Dirichlet inner product* of f and g is

$$(f, g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx$$

where dx denotes Lebesgue measure.

For a domain $D \subseteq \mathbb{C}$ with $\mathrm{diam}(\partial D) > 0$ (so that, e.g., ∂D does not consist of a single point), we let $H_0^1(D)$ be the Hilbert space completion of $C_0^\infty(D)$ with respect to $(\cdot, \cdot)_\nabla$.

12.2. **Properties of $(\cdot, \cdot)_\nabla$ and $H_0^1(D)$.** *Conformal invariance.* Suppose that $\varphi \colon D \to \widetilde{D}$ is a conformal transformation and $f, g \in C_0^\infty(D)$. Then

$$(f, g)_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla.$$

This calculation is part of Example Sheet 2. It implies that the map $H_0^1(D) \to H_0^1(\widetilde{D})$ given by $f \mapsto f \circ \varphi^{-1}$ is an isomorphism of Hilbert spaces.

Inclusion. Suppose that $U \subseteq D$ is open. If $f \in C_0^\infty(U)$, then trivially $f \in C_0^\infty(D)$. Therefore the inclusion map $\iota\colon H_0^1(U) \to H_0^1(D)$ is defined and associates $H_0^1(U)$ with a subspace of $H_0^1(D)$.

Orthogonal decomposition. Suppose that $U \subseteq D$ is open. Write $H_{\text{supp}} = H_0^1(U) \subseteq H_0^1(D)$ and H_{harm} for those functions $f \in H_0^1(D)$ which are harmonic in U. Then

$$H_0^1(D) = H_{\text{supp}} \oplus H_{\text{harm}}$$

is an orthogonal decomposition of $H_0^1(D)$.

To see orthogonality, suppose that $f \in H_{\text{supp}}$ and $g \in H_{\text{harm}}$. Then

$$\begin{aligned}(f,g)_\nabla &= \frac{1}{2\pi}\int \nabla f(x)\cdot\nabla g(x)dx\\ &= -\frac{1}{2\pi}\int f(x)\Delta g(x)dx \quad \text{(by integration by parts)}\\ &= 0.\end{aligned}$$

To see the final equality, note that f is equal to 0 outside of U and Δg is equal to 0 in U.

To see that these two spaces span $H_0^1(D)$, suppose that $f \in H_0^1(D)$ and let f_0 be the orthogonal projection of f onto H_{supp}. Then we want to show that $g_0 = f - f_0$ is in $H_{\text{harm}}(U)$. Suppose that $\varphi \in C_0^\infty(U)$. Then we have that

$$\begin{aligned}0 &= (\varphi, g_0)_\nabla \quad (\text{since } g_0 \perp H_{\text{supp}})\\ &= \frac{1}{2\pi}\int \nabla\varphi(x)\cdot\nabla g_0(x)dx\\ &= -\frac{1}{2\pi}\int \Delta\varphi(x)g_0(x)dx.\end{aligned}$$

On Example Sheet 2, it will be shown that this property implies that g_0 is harmonic and in fact C^∞ on U. (This is a special case of so-called *elliptic regularity.*)

12.3. **Gaussian free field definition.** In order to motivate our perspective on the GFF, we will begin with an aside about normal random variables on $\mathbb{R}^n$. Let

$$h = \alpha_1 e_1 + \cdots + \alpha_n e_n$$

where $\alpha_1, \ldots, \alpha_n$ are i.i.d. $N(0,1)$ and $e_1, \ldots, e_n$ is the standard basis on $\mathbb{R}^n$. Then h is a standard n-dimensional Gaussian.

If $x \in \mathbb{R}^n$, then

$$(h, x) = \sum_{j=1}^n \alpha_j x_j$$

is a $N(0, \|x\|^2)$ random variable. Also, if $x, y \in \mathbb{R}^n$, then $(h, x), (h, y)$ are jointly Gaussian with

$$\mathrm{cov}((h,x),(h,y)) = \sum_{j=1}^{n} x_j y_j = (x, y).$$

So, h is naturally associated with an entire family of Gaussian random variables (h, x) indexed by $x \in \mathbb{R}^n$ with mean zero and covariance given by the inner product on $\mathbb{R}^n$. (This is a simple of example of what is known as a "Gaussian Hilbert space", which refers to a family of Gaussian random variables indexed by the elements of a Hilbert space with covariance given by the inner product structure of the Hilbert space.)

Let (f_n) be an orthonormal basis (ONB) of $H_0^1(D)$ and let (α_n) be an i.i.d. $N(0, 1)$ sequence. Then the GFF h on D is defined by

$$h = \sum_{n=1}^{\infty} \alpha_n f_n.$$

Note that if $f = \sum_{n=1}^{\infty} \beta_n f_n \in H_0^1(D)$, then

$$(h, f)_\nabla = \sum_{n=1}^{\infty} \alpha_n \beta_n \sim N(0, \|f\|_\nabla^2).$$

Also, if $g = \sum_{n=1}^{\infty} \gamma_n f_n \in H_0^1(D)$, then $(h, f)_\nabla, (h, g)_\nabla$ are jointly Gaussian with

$$\mathrm{cov}((h,f)_\nabla,(h,g)_\nabla) = \sum_{n=1}^{\infty} \beta_n \gamma_n = (f, g)_\nabla.$$

So, the GFF is a family of Gaussian random variables $(h, f)_\nabla$ indexed by $H_0^1(D)$ with mean-zero and covariance given by the inner product on $H_0^1(D)$.

12.4. **Properties of the GFF.** *Conformal invariance.* If $\varphi \colon D \to \widetilde{D}$ is a conformal transformation, $h = \sum_{n=1}^{\infty} \alpha_n f_n$ is a GFF on D, then $h \circ \varphi^{-1} = \sum_{n=1}^{\infty} \alpha_n f_n \circ \varphi^{-1}$ is a GFF on $\widetilde{D}$ as $(f_n \circ \varphi^{-1})$ is an ONB of $H_0^1(\widetilde{D})$.

Markov property. If $U \subseteq D$ is open, then we can write $h = h_1 + h_2$, h_1, h_2 independent, h_1 a GFF on U, and h_2 harmonic on U.

To see that the Markov property holds, we note that we can take our ONB (f_n) of $H_0^1(D)$ in the definition of the GFF to consist of an ONB (f_n^1) of H_{supp} and an ONB (f_n^2) of H_{harm}. Then we have that

$$h = \sum_{n=1}^{\infty} \alpha_n^1 f_n^1 + \sum_{n=1}^{\infty} \alpha_n^2 f_n^2$$

where (α_n^1), (α_n^2) are independent i.i.d. sequences of $N(0, 1)$ random variables. The first summand is a GFF on U and the second summand is harmonic in U.

L^2 inner product of the GFF with a smooth function. By integration by parts, we have that

$$(h, f)_\nabla = -\frac{1}{2\pi}(h, \Delta f),$$

where the right hand side is the L^2 inner product of h with Δf. If $\varphi \in C_0^\infty(D)$, then we have that

$$\Delta^{-1}\varphi(x) = -\frac{1}{2\pi}\int G(x,y)\varphi(y)dy$$

where

$$G(x,y) = -\log|x-y| - \widetilde{G}_x(y)$$

and $\widetilde{G}_x$ is the harmonic function in D with boundary values given by $y \mapsto -\log|x-y|$ (so x here is fixed). Here, G is the so-called Green's function for Δ with Dirichlet boundary conditions on D. (If we want to emphasize the domain D on which G is defined we will write G_D for G.) Then (h,φ) is defined by setting

$$(h,\varphi) = -2\pi(h,\Delta^{-1}\varphi)_\nabla.$$

In particular, (h,φ) is a mean-zero normal random variable with variance

$$\begin{aligned}(2\pi)^2\|\Delta^{-1}\varphi\|_\nabla^2 &= (2\pi)^2(\Delta^{-1}\varphi,\Delta^{-1}\varphi)_\nabla\\ &= -2\pi(\Delta^{-1}\varphi,\Delta\Delta^{-1}\varphi) \quad \text{(by integration by parts)}\\ &= \iint \varphi(x)G(x,y)\varphi(y)dxdy.\end{aligned}$$

Similarly, if $\psi \in C_0^\infty(D)$, then (h,φ) and (h,ψ) are jointly Gaussian with

$$\mathrm{cov}((h,\varphi),(h,\psi)) = \iint \varphi(x)G(x,y)\psi(y)dxdy.$$

If $D = \mathbb{H}$, then we have that

$$G(x,y) = G_{\mathbb{H}}(x,y) = -\log|x-y| + \log|x-\overline{y}|.$$

Indeed, note that for $x \in \mathbb{H}$ fixed, the function $y \mapsto -\log|x-\overline{y}|$ is harmonic in $\mathbb{H}$ and has the same values on $\partial\mathbb{H}$ as the function $y \mapsto -\log|x-y|$.

Proposition 12.1. *The Green's function is conformally invariant. That is, if $D, \widetilde{D} \subseteq \mathbb{C}$ are domains and $\varphi\colon D \to \widetilde{D}$ is a conformal transformation, then*

$$G_D(x,y) = G_{\widetilde{D}}(\varphi(x),\varphi(y))$$

where G_D (resp. $G_{\widetilde{D}}$) denotes the Green's function on D (resp. $\widetilde{D}$).

Proof. Consider the function

$$\log|\varphi(x)-\varphi(y)| - \log|x-y| = \log\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|.$$

Since φ is a conformal transformation, it follows that it is a harmonic function in both x and y. It therefore follows that $G_D(x,y) - G_{\widetilde{D}}(\varphi(x),\varphi(y))$ is a harmonic function in both x and y. Therefore it suffices to show that $G_D(x,y) - G_{\widetilde{D}}(\varphi(x),\varphi(y))$ vanishes on ∂D. But this follows since G_D vanishes on ∂D and $G_{\widetilde{D}}$ vanishes on $\partial\widetilde{D}$. $\square$

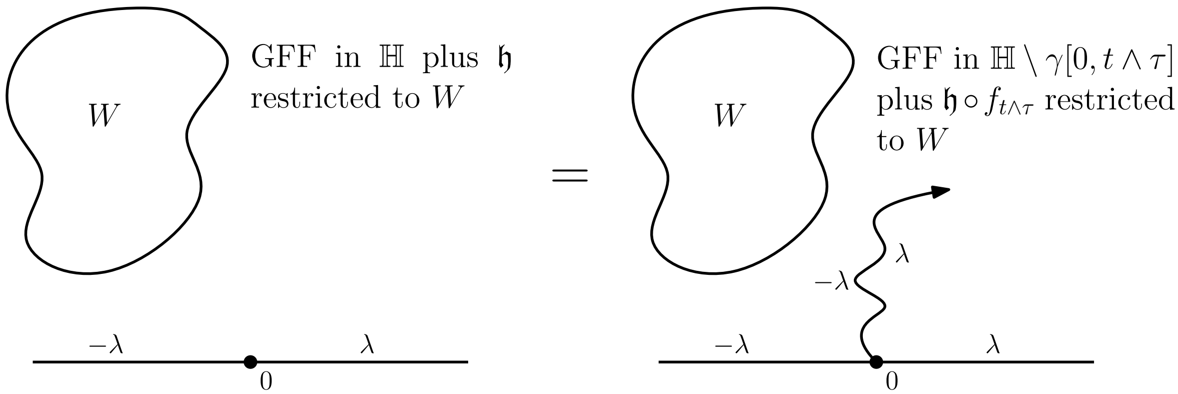

Figure 13.1. Illustration of the statement of Theorem 13.1, which gives two equivalent recipes for sampling from the law of a GFF on $\mathbb{H}$ plus the function which is harmonic in $\mathbb{H}$ with boundary conditions $-\lambda$ on $\mathbb{R}_-$ and λ on $\mathbb{R}_+$, restricted to W. The boundary values of the functions $\mathfrak{h}$ and $\mathfrak{h} \circ f_{t\wedge\tau}$ are shown. Note that the former is harmonic in $\mathbb{H}$ and the latter is harmonic in $\mathbb{H} \setminus \gamma[0, t \wedge \tau]$.

13. Level lines of the Gaussian free field

The GFF does not define a function. Rather, it is a random variable which takes values in the space of distributions. In a certain precise sense, it is *barely* not a function. When analyzing the GFF, one often pretends that it is a function and then asks what properties it might have which are analogous to those of a function. In this section, we will analyze the 0-level set of the GFF. That is, we will look at the set $\{x : h(x) = 0\}$ and show that it is closely connected to SLE_4. This connection is due to Schramm and Sheffield.

Theorem 13.1. *Let $\lambda = \pi/2$. Let γ be an SLE_4 in $\mathbb{H}$ from 0 to ∞, (g_t) be its Loewner evolution with driving function $U_t = \sqrt{\kappa} B_t = 2B_t$, and $f_t = g_t - U_t$. Let $W \subseteq \mathbb{H}$ open and let $\tau = \inf\{t \geq 0 : \gamma(t) \in W\}$. Let h be a GFF on $\mathbb{H}$ and $\mathfrak{h} = \lambda - \frac{2\lambda}{\pi} \arg(\cdot)$. Then we have that*

$$h \circ f_{t\wedge\tau} + \mathfrak{h} \circ f_{t\wedge\tau} \stackrel{d}{=} h + \mathfrak{h},$$

where the left and right sides are restricted to W.

See Figure 13.1 for an illustration of the statement of Theorem 13.1.

Before we proceed to the proof of Theorem 13.1, let us make a few remarks about the statement.

- The function $\mathfrak{h}$ is harmonic in $\mathbb{H}$ with boundary conditions given by $-\lambda$ on $\mathbb{R}_-$ and λ on $\mathbb{R}_+$.
- The function $\mathfrak{h} \circ f_{t\wedge\tau}$ is harmonic in $\mathbb{H} \setminus \gamma[0, t \wedge \tau]$ with boundary conditions $-\lambda$ on $\mathbb{R}_-$ and on the left side of $\gamma[0, t \wedge \tau]$ and λ on $\mathbb{R}_+$ and on the right side of $\gamma[0, t \wedge \tau]$.
- The theorem statement says that γ is a "ridge-line" of the GFF as the GFF is "constant" immediately to its left and right sides, and across γ it makes a jump of size 2λ.

- Theorem 13.1 is a simplified version of the theorem proved by Schramm and Sheffield, because we have stopped the SLE_4 upon hitting W and are only integrating it against functions which are supported in W.

Note that a random variable h taking values in $\mathbb{R}^n$ is a standard Gaussian if and only if for every $x \in \mathbb{R}^n$ we have that (h,x) is a $N(0,\|x\|^2)$ random variable. We will be applying the same principle here in the setting of the Gaussian distribution on $H_0^1(D)$ (i.e., the GFF) in order to prove Theorem 13.1.

To begin to prove Theorem 13.1, we fix $\phi \in C_0^\infty(W)$. We need to show that

$$(h \circ f_{t\wedge\tau} + \mathfrak{h} \circ f_{t\wedge\tau}, \phi) \stackrel{d}{=} (h + \mathfrak{h}, \phi).$$

In other words, we need to show that the left hand side is a $N(m_0(\phi), \sigma_0^2(\phi))$ random variable where

$$m_0(\phi) = (\mathfrak{h}, \phi) \quad \text{and} \quad \sigma_0^2(\phi) = \iint \phi(x) G_{\mathbb{H}}(x,y)\phi(y)dxdy.$$

This is equivalent to showing that

$$\mathbb{E}[\exp[i\theta(h \circ f_{t\wedge\tau} + \mathfrak{h} \circ f_{t\wedge\tau}, \phi)]] = \exp[i\theta m_0(\phi) - \frac{\theta^2}{2}\sigma_0^2(\phi)].$$

Let $(\mathcal{F}_t)$ be the filtration generated by U_t. Then we have that

$$\begin{aligned} &\mathbb{E}[\exp[i\theta(h \circ f_{t\wedge\tau} + \mathfrak{h} \circ f_{t\wedge\tau}, \phi)] \,|\, \mathcal{F}_{t\wedge\tau}] \\ =&\mathbb{E}[\exp[i\theta(h \circ f_{t\wedge\tau}, \phi)] \,|\, \mathcal{F}_{t\wedge\tau}] \exp(i\theta m_{t\wedge\tau}(\phi)). \end{aligned} \tag{13.1}$$

Here,

$$m_t(\phi) = (\mathfrak{h} \circ f_t, \phi)$$

and we have used above that $m_{t\wedge\tau}(\phi)$ is $\mathcal{F}_{t\wedge\tau}$-measurable. As the conditional law of $h \circ f_{t\wedge\tau}$ given $\mathcal{F}_{t\wedge\tau}$ is that of a GFF on $\mathbb{H}\setminus\gamma[0,t\wedge\tau]$ (by the conformal invariance of the GFF), it follows that (13.1) is equal to

$$\exp\left(i\theta m_{t\wedge\tau}(\phi) - \frac{\theta^2}{2}\sigma_{t\wedge\tau}^2(\phi)\right) \tag{13.2}$$

where

$$\sigma_t^2(\phi) = \iint \phi(x) G_t(x,y) \phi(y) dxdy$$
$$G_t(x,y) = G_{\mathbb{H}}(f_t(x), f_t(y)) = G_{\mathbb{H}}(g_t(x), g_t(y)).$$

Therefore to finish proving the theorem, it suffices to show that

$$\mathbb{E}\left[\exp\left(i\theta m_{t\wedge\tau}(\phi) - \frac{\theta^2}{2}\sigma_{t\wedge\tau}^2(\phi)\right)\right] = \exp\left(i\theta m_0(\phi) - \frac{\theta^2}{2}\sigma_0^2(\phi)\right).$$

To prove this, it suffices to show that the expression in (13.2) is a martingale. It in turn suffices to show that $m_t(\phi)$ is a martingale with quadratic variation process

$$[m.(\phi)]_t = \sigma_0^2(\phi) - \sigma_t^2(\phi)$$

because then (13.2) is the (complex) exponential martingale associated with $i\theta m_t(\phi)$.

We begin by checking that $m_t(\phi)$ is a martingale. To prove this, it suffices to show that

$$\begin{aligned}\mathfrak{h} \circ f_t(z) &= \lambda - \frac{2\lambda}{\pi} \arg(f_t(z)) \\ &= \lambda - \frac{2\lambda}{\pi} \mathrm{Im} \log(g_t(z) - U_t)\end{aligned}$$

is a martingale. It in turn suffices to show that $\log(f_t(z)) = \log(g_t(z) - U_t)$ is a continuous local martingale (as $\mathfrak{h} \circ f_t(z)$ is bounded). By Itô's formula, we have that

$$\begin{aligned}d\log(g_t(z) - U_t) &= \frac{1}{g_t(z) - U_t} d(g_t(z) - U_t) - \frac{1/2}{(g_t(z) - U_t)^2} d[U]_t \\ &= \frac{2}{(g_t(z) - U_t)^2} dt - \frac{1}{g_t(z) - U_t} dU_t - \frac{\kappa/2}{(g_t(z) - U_t)^2} dt \\ &= \frac{2 - \kappa/2}{(g_t(z) - U_t)^2} - \frac{1}{g_t(z) - U_t} dU_t \\ &= -\frac{1}{g_t(z) - U_t} dU_t,\end{aligned}$$

where we have used in the last line that $\kappa = 4$. This proves that $\mathfrak{h} \circ f_t(z)$ is a continuous local martingale. From its explicit form, we also have that

$$d[m.(\phi)]_t = \left(\iint \phi(z) \mathrm{Im} \left(\frac{2}{g_t(z) - U_t} \right) \mathrm{Im} \left(\frac{2}{g_t(w) - U_t} \right) \phi(w) dz dw \right) dt.$$

It is left to compute $d\sigma_t^2(\phi)$. Recall that

$$G_t(z, w) = G_{\mathbb{H}}(z, w) = -\mathrm{Re} \left(\log(g_t(z) - g_t(w)) - \log(g_t(z) - \overline{g_t(w)}) \right).$$

We will compute the Itô derivative of each of the two log expressions. The desired formula for $dG_t(z, w)$ will then follow by taking real parts. We have that

$$\begin{aligned}d\log(g_t(z) - g_t(w)) &= \frac{1}{g_t(z) - g_t(w)} \left(\frac{2}{g_t(z) - U_t} - \frac{2}{g_t(w) - U_t} \right) dt \\ &= \frac{-2}{(g_t(z) - U_t)(g_t(w) - U_t)} dt.\end{aligned} \tag{13.3}$$

We similarly have that

$$d\log(g_t(z) - \overline{g_t(w)}) = \frac{-2}{(g_t(z) - U_t)(\overline{g_t(w)} - U_t)} dt. \tag{13.4}$$

Combining (13.3) and (13.4), we see that

$$dG_t(z,w) = -\mathrm{Im}\left(\frac{2}{g_t(z)-U_t}\right)\mathrm{Im}\left(\frac{2}{g_t(w)-U_t}\right)dt.$$

Therefore

$$d\sigma_t^2(\phi) = -\left(\iint \phi(z)\mathrm{Im}\left(\frac{2}{g_t(z)-U_t}\right)\mathrm{Im}\left(\frac{2}{g_t(w)-U_t}\right)\phi(w)dzdw\right)dt.$$

This is exactly the form that we wanted, which completes the proof of the theorem.

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