**STOCHASTIC CALCULUS, LENT 2016, EXAMPLE SHEET 2**

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**Problem 1.** Suppose that $A, B \colon [0,\infty) \to \mathbb{R}$ are bounded and measurable and let $f \colon [0,\infty) \to \mathbb{R}$ be continuous and of finite variation. Show that

$$A \cdot (B \cdot df) = (AB) \cdot df$$

where $\cdot$ denotes the Lebesgue-Stieljes integral and $df$ is the Lebesgue-Stieljes measure associated with $f$.

**Problem 2.** Fix $p \geq 2$ and let $M$ be a continuous local martingale with $M_0 = 0$. Use Itô's formula, Doob's inequality, and Hölder's inequality to show that there exists a constant $C_p > 0$ such that

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |M_s|^p\right] \leq C_p \mathbb{E}[[M]_t^{p/2}].$$

**Problem 3.** Suppose that $f \colon [0,\infty) \to \mathbb{R}$ is a continuous function. Show that if $f$ has finite variation then it has zero quadratic variation. Conversely, show that if $f$ has finite and positive quadratic variation then it must be of infinite variation.

**Problem 4.** Let $B$ be a standard Brownian motion. Use Itô's formula to show that the following are martingales with respect to the filtration generated by $B$.

(1) $X_t = \exp(\lambda^2 t/2)\sin(\lambda B_t)$
(2) $X_t = (B_t + t)\exp(-B_t - t/2)$
(3) $X_t = \exp(B_t - t/2)$

**Problem 5.** Let $h \colon [0,\infty) \to \mathbb{R}$ be a measurable function which is square-integrable when restricted to $[0,t]$ for each $t > 0$ and let $B$ be a standard Brownian motion. Show that the process $H_t = \int_0^t h(s) dB_s$ is Gaussian and compute its covariance. (A real-valued process $(X_t)$ is Gaussian if for any finite family $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian).

**Problem 6.** Suppose that $f \colon [0,\infty) \to \mathbb{R}$ is continuous and of finite variation and that $X \colon [0,\infty) \to \mathbb{R}$ is bounded and left-continuous. Show that

$$\sum_{k=0}^{\lceil 2^n t \rceil - 1} X(k2^{-n})(f((k+1)2^{-n}) - f(k2^{-n}))$$

converges uniformly to $\int_0^t X(s) df(s)$ on each bounded interval of $[0,\infty)$.

**Problem 7.** Show that convergence in $(\mathcal{M}_c^2, \|\cdot\|)$ implies ucp convergence.

**Problem 8.** Show that the covariation $[\cdot,\cdot]$ is symmetric and bilinear. That is, if $M_1, M_2, M_3 \in \mathcal{M}_{c,loc}$ and $a \in \mathbb{R}$, then

$$[aM_1 + M_2, M_3] = a[M_1, M_3] + [M_2, M_3].$$

**Problem 9.** Let $B$ be a standard Brownian motion and let

$$\widehat{B}_t = B_t - \int_0^t \frac{B_s}{s} ds.$$

(1) Show that $\widehat{B}$ is not a martingale in the filtration generated by $B$.
(2) Show that $\widehat{B}$ is a martingale in its own filtration by showing that it is a Brownian motion. [Hint: show that $\widehat{B}$ is a continuous Gaussian process and identify its mean and covariance.]

**Problem 10.** Fix $d \geq 3$ and let $B$ be a Brownian motion in $\mathbb{R}^d$ starting at $B_0 = \overline{x} = (x, 0, \ldots, 0) \in \mathbb{R}^d$ for some $x > 0$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. For each $a > 0$, let $\tau_a = \inf\{t > 0 : \|B_t\| = a\}$.

(1) Let $D = \mathbb{R}^d \setminus \{0\}$ and let $h\colon D \to \mathbb{R}$ be defined by $h(x) = \|x\|^{2-d}$. Show that $h$ is harmonic on $D$ and that $M_t = \|B_t^{\tau_a}\|^{2-d}$ is a local martingale for all $a \geq 0$. Is $M$ a true martingale?
(2) Use the previous part to show that for any $a < b$ such that $0 < a < x < b$,

$$\mathbb{P}_{\overline{x}}[\tau_a < \tau_b] = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)}$$

where $\phi(u) = u^{2-d}$. Conclude that if $x > a > 0$, then

$$\mathbb{P}_x[\tau_a < \infty] = (a/x)^{d-2}.$$

**Problem 11.**

(1) Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where $(X, Y)$ is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale in $\mathbb{R}^2$. Show further that $M$ is a time-change of Brownian motion in $\mathbb{R}^2$.
(2) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and fix $z \in \mathbb{D}$. What is the hitting distribution for $Z$ on $\partial D$ in the case that $Z_0 = 0$? By applying a Möbius transformation $\mathbb{D} \to \mathbb{D}$ and using the previous part, determine the hitting distribution for $Z$ on $\partial\mathbb{D}$.

**Problem 12.** Let $N$ be a Poisson process of rate 1 and let $M_t = N_t - t$. Show that both $M_t$ and $N_t^2 - t$ are martingales. Explain why this does not contradict the Lévy characterization of Brownian motion.