STOCHASTIC CALCULUS, LENT 2016, EXAMPLE SHEET 1

Please send corrections to jpmiller@statslab.cam.ac.uk

Brownian motion

Problem 1. Let B be a standard Brownian motion. Show that $\widetilde{B}$ is also a standard Brownian motion in the case that:

(1) $\widetilde{B}_t = c^{-1}B_{c^2 t}$ for $c \neq 0$
(2) $\widetilde{B}_t = B_{t+a} - B_a$ for $a \geq 0$
(3) $\widetilde{B}_t = B_a - B_{a-t}$ for $t \in [0,a]$ and $a > 0$.

Problem 2. Let B be a standard Brownian motion. Prove that $B_t/t \to 0$ a.s. as $t \to \infty$. Show that the process $\widetilde{B}_t = tB_{1/t}$ for $t > 0$ is standard Brownian motion.

Problem 3 Let B be a standard Brownian motion. For $x \in \mathbb{R}$, let $T_x = \inf\{t > 0 : B_t = x\}$.

(1) Show that $T_x = x^2/Z^2$ where Z is a $N(0,1)$ random variable
(2) Show that the process $(T_x)_{x\geq 0}$ has independent and stationary increments. That is, for all $t_1 < \ldots < t_n$ the random variables $X_{t_{i+1}} - X_{t_i}$ are independent with law depending only on $t_{i+1} - t_i$.

Problem 4. Let B be a standard Brownian motion and fix $t \geq 0$. For $n \geq 1$, let $\Delta_n = \{0 : t_0(n) < t_1(n) < \cdots < t_{m_n}(n) = t\}$ be a partition of $[0,t]$ such that

$$\eta_n := \max_{1\leq i\leq m_n} \big(t_i(n) - t_{i-1}(n)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Show that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 = t$$

in L^2. Show that if the subdivision is dyadic, then the convergence is also almost sure.

Problem 5. Let B be a standard Brownian motion and let $\mathcal{Z} = \{t \geq 0 : B_t = 0\}$ be the set of times at which B is equal to 0.

(1) Show that $\mathcal{Z}$ is closed.
(2) Show that the Lebesgue measure of $\mathcal{Z}$ is almost surely equal to 0 [Hint: use Fubini's theorem]

Finite variation and previsible processes

Problem 6. Let H be a previsible process. Show that H_t is $\mathcal{F}_{t^-}$-measurable for all $t \in (0,\infty)$, where $\mathcal{F}_{t^-} = \sigma(\mathcal{F}_s : s < t)$.

Problem 7.

(1) Suppose that $(X^n)_{n\geq 1}$ is a sequence of cádlág functions such that $X^n \to X$ uniformly on $[0,t]$. That is, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $\sup_{s\in[0,t]} |X_s^n - X_s| < \epsilon$. Show that X is cádlág on $[0,t]$.
(2) Suppose that X is cádlág and let V be its total variation. Assume that $V(t) < \infty$ for some fixed $t > 0$. Show that $V^n \to V$ uniformly on $[0,t]$ and deduce that V is cádlág on $[0,t]$.

Problem 8. Let $(\mathcal{F}_t)$ be a filtration, let T be a stopping time, and let

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \le t\} \in \mathcal{F}_t \quad \forall t \ge 0\}.$$

(1) Show that $\mathcal{F}_T$ is a σ-algebra.
(2) Show that T is $\mathcal{F}_T$-measurable.
(3) Suppose that X is a cádlág, adapted process. Show that X_T is $\mathcal{F}_T$-measurable.

Problem 9. Suppose that B is a standard Brownian motion.

(1) Let $T = \inf\{t \ge 0 : B_t = 1\}$. Show that H defined by $H_t = \mathbf{1}\{T \ge t\}$ is previsible.
(2) Let

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if} \quad x \le 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Show that $(\operatorname{sgn}(B_t))_{t \ge 0}$ is a previsible process which is neither left nor right continuous.

Problem 10. Suppose that $f \colon [0,\infty) \to \mathbb{R}$ is absolutely continuous, in the sense that

$$f(t) = f(0) + \int_0^t f'(s)ds \quad \text{for all} \quad t \ge 0$$

for an integrable function f'. Let $V(t)$ be the variation of f on $[0,t]$. Show that

$$V(t) = \int_0^t |f'(s)|ds.$$

Local martingales and quadratic variation

Problem 11. Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family $\mathcal{X} = \{\mathbb{E}[X \,|\, \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \quad \text{is a } \sigma\text{-algebra}\}$ is UI.

Problem 12. Suppose that M is a continuous local martingale with $M_0 = 0$. Show that M is an L^2-bounded martingale if and only if $\mathbb{E}[[M]_\infty] < \infty$.

Problem 13. Let X be a continuous local martingale. Show that if $\mathbb{E}[\sup_{0 \le s \le t} |X_s|] < \infty$ for each $t \ge 0$ then X is a martingale.

Problem 14.

(1) Suppose that M, N are independent continuous local martingales. Show that $[M,N]_t = 0$. In particular, if $B^{(1)}$ and $B^{(2)}$ are the coordinates of a standard Brownian motion in $\mathbb{R}^2$, this shows that $[B^{(1)}, B^{(2)}]_t = 0$ for all $t \ge 0$.
(2) Let B be a standard Brownian motion in $\mathbb{R}$ and let B be a stopping time which is a.s. not constant. By considering B^T and $B - B^T$, show that the converse to the previous part is false. Hint: show that T is measurable with respect to the σ-algebras generated by both B^T and $B - B^T$.