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Problem 1. Suppose that \( \gamma: [0, T] \to \overline{H} \) is a simple curve (i.e., \( s \neq t \) implies \( \gamma(s) \neq \gamma(t) \)) with \( \gamma(0) = 0 \) and \( \gamma(t) \in \overline{H} \) for all \( t \in (0, T) \). Show that \( A_t = \gamma((0, t]) \) for \( t \in [0, T] \) is a family of locally growing compact \( H \)-hulls. Show, moreover, that there exists a homeomorphism \( \phi: [0, T] \to [0, \frac{1}{2} hcap(A_T)] \) so that \( hcap(A_{\phi^{-1}(t)}) = 2t \) for all \( t \in [0, \frac{1}{2} hcap(A_T)] \). (This is the so-called capacity parameterization of \( \gamma \).)

Problem 2. Suppose that \( U: [0, T] \to \mathbb{R} \) is a continuous function. Let \( g_t(z) \) solve the chordal Loewner equation

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.
\]

Show for each \( t \in [0, T] \) that \( g_t \) is a conformal transformation from its domain onto \( \overline{H} \) with \( g_t(z) \to z \to 0 \) as \( z \to \infty \) using the following steps.

- Show that \( t \mapsto \text{Im}(g_t(z)) \) is decreasing in \( t \), hence for each \( z \in \overline{H} \), \( t \mapsto g_t(z) \) is defined up until \( \tau_z = \sup \{ t \geq 0 : \text{Im}(g_t(z)) > 0 \} \). Conclude that \( H_t = \{ z : \tau_z > t \} \) is the domain of \( g_t \).
- Show for each \( t \in [0, T] \) that \( z \mapsto g_t(z) \) is complex differentiable on \( H_t \).
- Show for each \( t \in [0, T] \) that \( z \mapsto g_t(z) \) has an inverse defined on \( \overline{H} \) by showing that \( g_t(f_t(w)) = w \) for all \( w \in \overline{H} \) where \( f_s \) for \( s \in [0, t] \) solves the so-called reverse chordal Loewner equation

\[
\partial_s f_s(w) = \frac{2}{f_s(w) - U_{t-s}}, \quad f_0(w) = w.
\]

Problem 3. Suppose that \( U_t = \sqrt{\kappa} B_t \) where \( B \) is a standard Brownian motion and let \( (g_t) \) solve

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.
\]

- (Markov property) Suppose that \( \tau \) is a stopping time for \( U \) which is almost surely finite and let \( g_t = g_{\tau+t}(g_\tau^{-1}(z + U_\tau)) - U_t \). Show that the maps \( (\tilde{g}_t) \) have the same distribution as the maps \( (g_t) \).
- (Scale invariance) Fix \( \tau > 0 \) and let \( \tilde{g}_t(z) = r g_{t/r^2}(z/r) \). Show that the maps \( (\tilde{g}_t) \) have the same distribution as the maps \( (g_t) \).

Suppose that \( D \) is a simply connected domain, \( x, y \in \partial D \) are distinct, and \( \varphi: \overline{H} \to D \) is a conformal transformation with \( \varphi(0) = x \) and \( \varphi(\infty) = y \). Explain why the definition of \( \text{SLE}_\kappa \) given by \( \varphi(\gamma) \) where \( \gamma \) is an \( \text{SLE}_\kappa \) in \( \overline{H} \) from 0 to \( \infty \) is well-defined.

Problem 4.

- Suppose that \( B \) is a standard Brownian motion and \( a < 0 \). Show that \( \sup_{t \geq 0} (B_t + at) < \infty \) almost surely.
- Suppose that \( (g_t) \) is the family of conformal maps which solve the Loewner equation with driving function \( U_t = \sqrt{\kappa} B_t \) and, for each \( x \in \mathbb{R} \), let \( V_t^x = g_t(x) - U_t \) and \( \tau_x = \inf \{ t \geq 0 : V_t^x = 0 \} \). For each \( 0 < x < y \), let \( g(x, y) = \mathbb{P}[\tau_x = \tau_y] \). Show that if \( g(1, 1 + \epsilon/2) > 0 \) for all \( \epsilon \in (0, \epsilon_0) \) for some \( \epsilon_0 > 0 \) then \( g(x, y) > 0 \) for all \( 0 < x < y \).
Problem 5. Fix $T > 0$ and let $D \subseteq \mathbb{H}$ be a simply connected domain. Suppose that $(A_t)_{t \in [0,T]}$ is a non-decreasing family of compact $\mathbb{H}$-hulls which are locally growing with $A_0 = \emptyset$, $hcap(A_t) = 2t$ for all $t \in [0,T]$, and $A_T \subseteq D$. Let $\psi : D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Show that the family of compact $\mathbb{H}$-hulls $\tilde{A}_t = \psi(A_t)$ for $t \in [0,T]$ is locally growing with $A_0 = \emptyset$ and with

$$hcap(\tilde{A}_t) = \int_0^t 2(\psi'(U_s))^2 ds \quad \text{where} \quad \psi_t = \tilde{g}_t \circ \psi \circ g_t^{-1} \quad \text{for each} \quad t \in [0,T]$$

and $\tilde{g}_t$ is the unique conformal transformation $\mathbb{H} \setminus \tilde{A}_t \to \mathbb{H}$ with $\tilde{g}_t(z) - z \to 0$ as $z \to \infty$.

Problem 6. In the setting of the previous problem, show that

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi''(U_t).$$

Problem 7. Suppose that $(A_t)$ is a non-decreasing family of $\mathbb{H}$-hulls which are locally growing and with $A_0 = \emptyset$. For each $t \geq 0$, let $a(t) = hcap(A_t)$ and assume that $a(t)$ is $C^1$. For each $t \geq 0$, let $g_t$ be the unique conformal transformation which takes $\mathbb{H} \setminus A_t$ to $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Show that the conformal maps $(g_t)$ satisfy the ODE:

$$\partial_t g_t(z) = \frac{\partial_t a(t)}{g_t(z) - U_t}, \quad g_0(z) = z$$

for some continuous, real-valued function $U_t$. (Hint: perform a time-change so that the hulls are parameterized by capacity, apply Loewner’s theorem as proved in class, and then invert the time change.)

Problem 8. Suppose that $B$ is a standard Brownian motion starting from $B_0 = x > 0$. For each $a \in \mathbb{R}$, let $\tau_a = \inf\{t \geq 0 : B_t = a\}$.

- For $a < x < b$, explain why $\mathbb{P}[\tau_a < \tau_b] = (b-x)/(b-a)$.
- Using the Girsanov theorem, explain why the law of $B$ weighted by $B_{\tau_a \wedge \tau_b}$ is equal to that of a BES$^3$ process stopped upon hitting $b$. That is, if $\mathbb{P}$ denotes the law of $B$ and we define the law $\tilde{\mathbb{P}}$ using the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \begin{bmatrix} B_{\tau_a \wedge \tau_b} \\ E[B_{\tau_a \wedge \tau_b}] \end{bmatrix}$$

then the law of $B$ under $\tilde{\mathbb{P}}$ is that of a BES$^3$ process stopped upon hitting $b$.
- Explain why a standard Brownian motion conditioned to be non-negative is a BES$^3$ process.
- More generally, explain why a BES$^d$ process with $d < 2$ conditioned to be non-negative is a BES$^{4-d}$ process.

Problem 9. Suppose that $(g_t)$ is the family of conformal maps associated with an SLE$_\kappa$ with driving function $U_t$, i.e., $U_t = \sqrt{\kappa}B_t$ where $B$ is a standard Brownian motion. Fix $z \in \mathbb{H}$ and let $z_t = x_t + iy_t = g_t(z)$. Assume that $\rho \in \mathbb{R}$ is fixed. Use Itô’s formula to show that

$$M_t = |g_t'(z)|^{(8-2\kappa + \rho)/8\kappa} g_t^2/8\kappa |U_t - z_t|^{\rho/\kappa}$$

is a continuous local martingale. (Hint: let

$$Z_t = \frac{(8 - 2\kappa + \rho)\rho}{8\kappa} \log g_t'(z) + \frac{\rho^2}{8\kappa} \log y_t + \frac{\rho}{\kappa} \log(U_t - z_t),$$

compute $dZ_t$ using Itô’s formula, take its real part, and exponentiate.)
**Problem 10.** Assume that we have the setup of Problem 9. Let $\Upsilon_t = y_t / |g'_t(z)|$. You may assume that

$$\frac{1}{4} \leq \frac{\Upsilon_t}{\text{dist}(z, \gamma([0,t]) \cup \partial H)} \leq 4.$$

- Let $S_t = \sin(\arg(z_t - U_t))$. Explain why

$$M_t = |g'_t(z)|^{(8-\kappa+\rho)/(4\kappa)} \Upsilon_t^{\rho/(\kappa+8)} S_t^{-\rho/\kappa}.$$

- By considering the above martingale with the special choice $\rho = \kappa - 8$, show that if $\kappa > 8$ then the SLE$_\kappa$ curve $\gamma$ almost surely hits $z$. Conclude that $\gamma$ fills all of $H$. (Hint: recall that we showed in class that $\gamma$ fills $\partial H$. Deduce from this and the conformal Markov property that $\gamma$ cannot separate $z$ from $\infty$ without hitting it. Consider the behavior of $S_t$ when $\gamma$ is hitting a point on $\partial H$ with either very large positive or negative values.)